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### By S. EDWARD WARREN, C.E.

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# PRIMARY GEOMETRY,

WITH

SIMPLE AND PRACTICAL EXAMPLES IN PLANE AND PROJECTION DRAWING, AND SUITED TO ALL BEGINNERS.

BY

S. EDWARD WARREN, C.E.,

FORMERLY PROFESSOR IN THE RENSSELAER POLYTECHNIC INSTITUTE, ETC.,
AND AUTHOR OF A SERIES OF ELEMENTARY AND HIGHER
TEXT-BOOKS ON DESCRIPTIVE GEOMETRY
AND INDUSTRIAL DRAWING.

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*Geometry should be begun as early and as simply in behalf of industrial life as Arithmetic is in behalf of business life.*

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# PREFACE.

A LARGE portion of the many and varied pursuits of life consists of the mechanical industries. This is made easily yet strikingly evident by the number and variety of the advertisements in even a single number of any one of the many periodicals devoted to such industries. The same fact is otherwise forced upon one's attention by an examination of the statistics of industry, whether local or general.

Upon reflection, it occurs to the mind that the immediate object of all these mechanical industries is to change the *form* or the *place* of matter. When a block of marble is hewn into a section of a cornice, its *form* is changed; when the piece is raised to the top of the wall, its *place* is changed. In both operations there must be a consideration of points, lines (of shape or movement), angles, surfaces, and solids. But the study of just these things, as to the principles which apply in all mechanical operations, is the study of Geometry. Hence we may confidently say at once, that Geometry should be begun as early and as simply, in behalf of industrial life, as arithmetic is in behalf of business life.

The object of this volume is, accordingly, to contribute to a general earlier beginning of the study of geometry; in the belief, moreover, that without such beginning, elementary education is somewhat one-sided.

The means to the end proposed, consist in such a treatment of the subject as shall make an earlier beginning of it possible;

that is, both the matter and the form have been made simpler than has hitherto been usual.

Again: besides the fact that geometry goes hand in hand with arithmetic in all of the multitude of pursuits, from the humblest to the highest, in which *form*, *size*, and *place*, as well as quantity and cost, are to be attended to, it should be never forgotten, that while arithmetic deals with figures, or the *signs of things*, geometry deals with *things themselves*, — points, lines, angles, surfaces, and solids. Hence, having regard to a child's natural tendency to deal directly with realities rather than with abstractions, a properly balanced education requires that geometry should at least accompany, if not precede, arithmetic in a child's proper mental training.

It hardly needs to be added, that geometrical models and other suitable handy objects should be ready in all practicable abundance to accompany the study.

With regard to details: —

1°. It has seemed best not to confuse the work with too great a variety of applications.

2°. Numerous examples for practice have been introduced at short intervals, to lend interest to the subject, show its usefulness, and test the pupils' understanding of the principles.

3°. The truths of *form*, as needed in *drawing*, have been made prominent, while not neglecting elementary ones of *measure*.

With these explanations, this brief volume is committed to the hands of teachers, in hope that it may, in behalf of both mental and industrial interests, lead to a wide increase of the much-needed early attention to geometry which is now beginning to appear.

NEWTON, MASS., *June*, 1887.

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# A PRIMARY GEOMETRY.

## CHAPTER I.

## INTRODUCTION.

### GEOMETRY AND OTHER MATHEMATICS.

**1.** We have all heard of **Mathematics**, and have an idea that it means the study of reckoning and measuring.

We have also, perhaps, all heard of *Arithmetic*, *Algebra*, and *Geometry* as branches of Mathematics. We think of *Arithmetic* as ciphering with figures, or working with figures to find numbers. If we know any thing at all of *Algebra*, perhaps we think of it as ciphering with letters instead of figures, or with a mixture of letters and figures, in order to work out some kinds of examples which cannot be done with figures alone.

We may also have an idea that *Geometry* is quite different from Arithmetic or Algebra, and that it is a kind of study of lines and angles, and of figures and bodies bounded or marked by lines and angles.

All these ideas are very good so far as they go. But let us get a little fuller and clearer idea of some of these subjects. First of all, we shall need to know what *quantity* means.

**2.** In common language, quantity means any thing that can be more or less.

In mathematics, quantity means whatever can be measured.

**3.** For example, if one boy is healthier or wiser or happier than another, no one can weigh or measure, or write down in figures, how much healthier or wiser or happier one boy is than the other. But if one boy is heavier or taller or older than the other, we can find out, and write in figures, the difference in their weights or heights or ages.

Also, if we know how much each one of ten boys weighs, we can find out what they all weigh. Or, if at night I know how much money I had in the morning, and how much I had at night, I can find the difference, or how much I have spent, lost, or given away during the day. Or, again, if I know just what my food costs in one week, I can find out what it will cost, at the same rate, for twenty or fifty weeks. And, finally, if I have a hundred apples or cakes to give to a party of children, I can find out how many apples or cakes each child will have, or how many children can have a certain number of apples apiece.

All these simple illustrations enable us to understand that Mathematics is the study of reckoning and measuring all such quantities as can be reckoned and measured.

The same illustrations also show us something more of the uses of *Arithmetic*. This study is always useful in what we call *business-life;* that is, in keeping account of gains and losses, cost, expense, buyings and sellings, receivings and givings, etc.: also in the summing up, taking away, repeating, and dividing or sharing of all sorts of quantities that can be measured and expressed in numbers.

4. Now, leaving arithmetic and algebra, let us proceed to make a good beginning in learning something of what geometry is, and something of what it is useful for.

First, we see all about us, both in nature and in every-day life, at home or in the street, a great many objects. Some of these, as a heap of earth, a tree, a bag of feathers or of meal, or a rough and ragged rock, have no regular shape that we can clearly describe in words, or that we can measure with a foot-rule. Other objects, as a door, a soap-box, a tin pail, a tin fruit-can, a ball, a tower, a spire, a wheel, a board, a brick tile, a fenced field, and many others, have simple regular shapes which can be easily described in words, and measured.

5. Now, **Geometry** is the study of the *forms* only of such regular objects as can be exactly described and measured. Geometry is not concerned with the color, hardness, material, etc., of bodies, but only with their forms; and, indeed, with forms with the least possible thought of any bodies to which they belong.

6. But to make Geometry interesting, as well as to have some idea of what it is, we must see some use in knowing something of it. Let us, then, next learn some of the ways in which it may be useful. We *know*, by looking about in city or country, that the store-keeper, the mechanic, the laborer, the banker, and many others, all have to keep accounts, or make some other use of figures, and must therefore have a knowledge of *arithmetic.* Now, to whom is *geometry* useful, and for what?

Look again, and see the carpenter, the mason, the wheelwright, the tinsmith, the machinist, the watchmaker, the cabinet-maker, the founder, the surveyor, engineer, architect, organ-builder, cooper, paper-box maker, brass-worker, draughtsman, sea-captain, and others, all of whom have to use tools to make things, or instruments to observe and mark things, of certain exact *sizes*, of certain *forms*, and fitted or placed in certain *positions*. It is therefore very plain that geometry, which teaches us all about forms and the measuring of them, must be useful to all of these.

7. But, besides the usefulness of geometry in many of the practical pursuits of life, it is highly useful in *training the mind to think exactly* about the things of which it treats. This is a great advantage; for, if the mind thinks exactly on any one subject, it will be more likely to do so on all others.

With the good reasons now given for being interested in geometry, let us proceed to learn more of it.

**Practice.** — [This is *thinking* or *doing* for yourself, first, what you have read or been taught; or, second, other things which you have learned enough to enable you to think or do.]

1. What is mathematics about?
2. What are some of its branches, and what is each about?
3. What does quantity mean in common speech and in mathematics?
4. Give examples of quantities that can be measured, and of some that cannot.
5. Mention some objects that have a shape that cannot be exactly described and measured, and some others that can.
6. Which of these is geometry concerned with?
7. Mention any purpose for which arithmetic is useful, and for each one of them mention one for which geometry is useful.
8. All to whom geometry is useful, work to accomplish what three purposes?

8. Here (Fig. 1) are a square block and a round ball, which just balance each other. Supposing them to be of the same material, as cast-iron, they must be of the same size; that is, they both contain the same quantity of iron. Thus we see that bodies

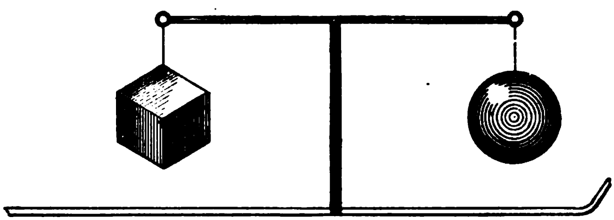

Fig. 1.

may have the *same size*, but quite *different forms*. Also, two equal balls of clay may be moulded, the one into the figure of a horse, the other into a square tile, and may thus be *equal in size*, but *different in form*.

9. Again: consider Bunker-hill Monument, 230 feet high, and then a stone post of the same shape, 4 feet high; or, suppose a row of casks, or of mugs or flower-pots, all of each of these articles of the same shape, but of different sizes. All these show that bodies of the *same form* may have very *different sizes*. Also a marble and the moon are of the same *form*, but very different in *size*.

10. Once more: besides the *form* and the *size* of a body, it is often important to notice its *position*, or the way in which it is placed. Thus the positions of the doors and windows in a room affect the position of the furniture in the room; the different positions of the earth at different times of year affect the manner in which the sun's rays, which are straight, strike it, and so make the difference between winter and summer; and a wooden roller lying on a floor will easily roll, but, if standing on its flat end, will remain still.

11. From the illustrations given in the last three articles, we see that *form*, *size*, and *position* are the three root-ideas out of which geometry grows.

Of these three, the *form* of a body or a figure gives it its name, as a ball or sphere, a square, or a circle; its *size* means how large it is, or how much room it takes up; and its *position* is where it is, or how it is placed.

Accordingly *geometry is the study of regular bodies or figures* so as to learn the many beautiful and useful truths about their *sizes*, *forms*, and *positions*, and then to apply these truths to the business of *measuring their size* and of *fitting them to each other in form and position*.

**Practice.** — 1. How do a square block and a round ball of equal weight and the same material differ?

2. Mention other examples of equal size and different form.

3. How do a marble and a base-ball differ?

4. Mention other objects of different size, but the same form.

5. Which gives the *name* to a body, — its form, or its size?

6. What is the *position* of a body?

7. Does difference of position change the size, form, or name of a body?

8. Give some examples of the influence of the position of a body upon the size, form, or position of another body.

9. How would you arrange bricks so as to provide for storing water? How to provide for the flow of water? How to support a floor?

10. By what means are the truths of geometry learned?

11. To what generally useful art does the study of geometry lead?

### Space.

**12.** Think of the place where you stand. Think of the places next to you on every side. Think of the places next beyond each of these. Keep on thinking in this way till your thoughts reach the stars. Then think, as you watch the stars on a clear night, of other stars filling all the places between the stars that you see, till the night sky would become an unbroken sheet of stars. Then think of another such sheet of stars beyond those, and of another beyond those, and so on without end. *Then* you will have thought what space is.

**Space** *is everywhere* where any thing is or might be.

**Space** *is the sum of all possible places or positions.*

Space has neither beginning nor end, and it contains every thing.

**13.** When any thing has no beginning or end, it has no boundaries, and extends beyond all possible measure. It is then called infinite. Space is infinite, or extends infinitely every way from whatever place in it we can think of.

**14.** In practice, the word "space" is used in various senses. When we say that we took a walk and did not meet any one for the space of two miles, we mean a *distance* or *length* of two miles on the road, or a *line* two miles long. When we speak of the open space around the house, we mean an *area* or *surface* of ground where one can move about. When a man has more goods than he can pack into a room, and says he wants more space, he may mean more floor space, or he may mean that he wishes that the room were larger every way. Finally, when we say that the sun, moon, and stars are situated in space, we mean space that extends everywhere without limit.

The other kinds or forms of space mentioned — the distance, the door-yard or floor, and the room — have ends, limits, or bounds. We call them limited portions of space.

All bodies or forms occupy limited portions of space. We shall next wish to learn how the way in which they occupy space is expressed.

### Direction and Extension.

**15.** At whatever point in space we may be, as soon as we think of any other point, we naturally ask two questions. These are, "Which way?" and "How far?"

The **direction** of the second point from the first answers the question, *Which way?*

Its **distance** answers the question, *How far?*

**16. Direction** is aim, tendency, or pointing at any point *towards some other point;* as of a weather-vane pointing north, east, etc., or of a rifle-ball aiming towards a mark, or of a cannon-ball rolling on smooth ice, and so tending to move towards some one point. Direction is named in various ways according to convenience, as north, south, forward, backward, right, left, up, down, etc.

There are two principal kinds of direction which are most familiar, and with which other directions are therefore compared. We will attend to these first.

**17.** *When the point aimed at is neither above nor below the point left;* as, when I roll a marble on the floor directly into a corner of the room, the direction is called *horizontal. Any* direction on a floor or frozen surface of a pond, or on any *level* surface, is horizontal.

**18.** When the point aimed at is directly overhead or under foot, the direction is *vertical.* A stone dropping down a well marks a vertical direction; so does a string to which a weight is hung, as a curtain-cord with a heavy tassel, or a well-rope.

We thus see that there may be *many horizontal directions* at any one point, but only *one vertical direction* at that point.

**19.** All other directions, which are neither horizontal nor vertical, are called *inclined* or *slanting*. Such are lines from top to bottom of the roof of a house.

**20. Extension** is a portion of space limited in all *directions* at certain *distances* from some starting-point.

Among all the different directions on which certain distances serve to mark the limits of any given kind of extension or portion of space, three are of chief importance, as will presently be explained.

**Practice.** — 1. If the whole earth were instantly blotted out of existence, and you left alive where you are, in what would you be?

2. In how many directions could you then see?

3. How far would space extend in every direction from you?

4. When a gardener speaks of the spaces between the trees of a row, what kind of spaces does he mean?

5. If a large open space outside the village is spoken of, in what sense is space used?

6. When you say, "Go north," "Go to the right," "Climb up," what thought are you expressing?

7. When you are at one point, and facing some other point, what do you call the way you are facing?

8. When asking, "Which way?" what is it that you are thinking of?

9. What is a *horizontal* direction? Give examples.

10. When is a direction vertical?

11. How many horizontal directions are there at any one point?

12. How many vertical ones?

13. How would you place a carriage-wheel so that all its spokes should be horizontal?

14. When the wheel is in its proper position on the carriage, how many of its spokes are vertical?

15. Mention other examples of horizontal and vertical lines.

16. What are inclined lines? Mention some examples.

17. What is extension?

**21.** Here is a broad field (Fig. 2). So long as a boy runs on the north and south line, NS, he will make no progress in an east and west direction; that is, he will be no farther east or west than when he started. Also, so long as he runs on the east and west line, EW, he will make no progress in a north and south direction.

And, finally, if he were to go *straight up*, or *vertically*, in a balloon on the line AB, he would get no farther north or south, or east or west, than he was at A.

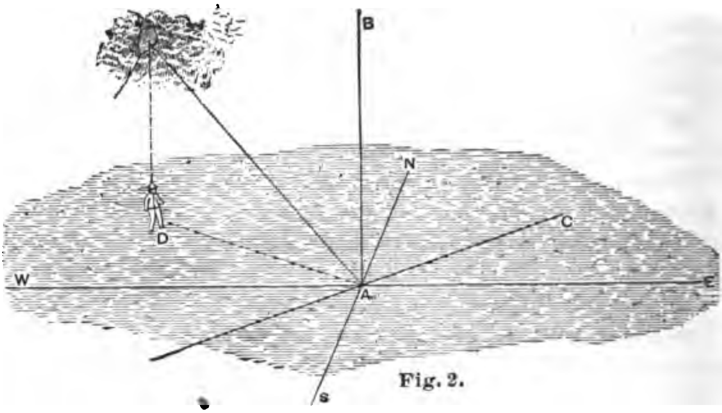


Fig. 2.

From this we see that *we can always take three straight lines through one point, as A, and in such a manner, that, in moving along any one of them, no advance is made in the direction of either of the others.*

The three lines which meet at one corner of a room, as AB, AC, AD (Fig. 3), are an example of such lines; and so are the three which meet at any one corner of a square box or of a brick.

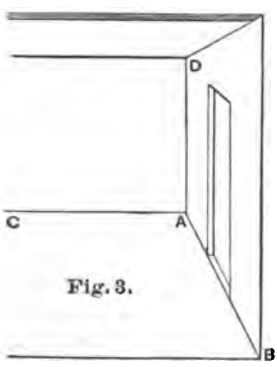


Fig. 3.

**22.** But we will return to our field again.

If a boy starts out from A on a north-east line, as AC, he will, as the very name of the line shows, be getting farther north, and farther east too, the farther he goes. Likewise the boy at D on the north-west line AD is farther north and also farther west than at A. Finally, the kite directly overhead to the boy at D is farther *north*, farther *west*, and farther *up*, than the point A; and the like can be said of any line through A besides the lines NS, EW, and AB.

**23.** This shows that *three, and only three, fixed straight lines can be so placed, with respect to each other, at any given point, that, in moving on any one of them, we shall make no progress in the direction of either of the other two.*

1°. Each of such lines is said to be *perpendicular* to each of the other two.

2°. Each of such lines is also said to be *at right angles* to each of the other two.

3°. As there is only *one* such kind of position which three straight lines can have with respect to each other, their position forms a standard with which to compare all other directions.

4°. All other lines are partly in the direction of two or of all three of these three lines, as shown above (22). But no one of these three lies at all in the direction of either of the others. Each is therefore said to be independent of the others, or their directions may be called *three independent directions.*

**24.** The three independent directions may be well represented by three straight wires firmly fastened together in the directions of those three edges of a square-cornered block which meet at any one of the corners, as at A in Fig. 4. As indicated also by the letters, one of the wires, FB, lies directly in a forward and backward direction; another, RL, extends from right to left; and the third, UD, directly up and down. Or we may think of the same wires as lying exactly north and south, east and west, and up and down or vertically.

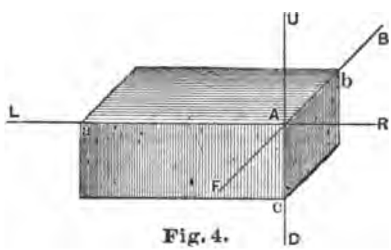

Fig. 4.

**25.** *Extension in any direction, when expressed in a certain measure*, as feet, yards, meters, etc., is called a *dimension.* The measures of any form of extension or portion of space in the three directions now explained are the ones usually understood by its dimensions. Separately, they are called *length*, *breadth*, and *thickness.* Thus, in Fig. 4, A*a* is the length, A*b* the width, and Ac the thickness, of the square-cornered block there shown.

Breadth is sometimes called width, and thickness is sometimes called *height*, as in the case of a building or a room; and sometimes it is called *depth*, as in case of a well. *But note* that any dimension may sometimes be called depth; for, when a hole is bored into a timber, or drilled into a rock, we speak of the *depth* of the hole, in whatever direction it is made.

**Practice.**—1. If a mouse runs along the edge of the floor, on the *length* of a square-cornered room, from a corner A to the next corner, B, does he at the same time make any progress in the direction of the *width* of the room?

2. If, next, he runs, in like manner, to the next corner, C, of the floor, what dimension of the room does he run over?

3. *While* he is running the *width* of the room, does he make any progress in the direction of its *length?*

4. How else could he run from A to C?

5. When he runs directly from A to C, he progresses at each step in the direction of what dimensions of the room?

6. Suppose D to be the corner of the ceiling directly over C, and the mouse able to run up the corner of the room, would he, when running up the *height* of the room, make any progress in the direction of the *length* or the *width* of the room?

7. But suppose him to run along a pole lying against the wall from the corner B of the floor to the corner D over the next corner, C, of the floor; then, at each step, he would be progressing in the direction of what dimensions of the room?

8. Finally, let a pole be placed from the first corner, A, to the farthest upper corner, D, over C; then, when the mouse runs up or down this pole, he makes progress in the direction of how many dimensions of the room?

9. How many dimensions has the room such that, when moving on each, we do not make any progress in the direction of any of the others?

10. When the mouse is running *up* the last-named pole from A, how many and which surfaces of the room is he getting farther from at every step?

11. Which ones is he getting nearer to?

12. When he runs straight from A to the next corner, B, how many surfaces of the room is he getting nearer to at each step?

13. How many when he is running from B to C, the *width?*

14. How many when he is running up the height from C to D?

15. The *measures* of any body in three such directions as are mentioned in Question 9 are called what?

16. What *position* are such directions said to have to each other?

17. What *angle* are they said to make with each other?

18. Name the *three dimensions*, and how these names sometimes vary.

19. Bind three wires or slender rods together in the three independent directions.

20. Give examples of the use of the words "length, breadth, and thickness."

21. Also of the use of the words "length, width, and height."

22. Also of the words "length, breadth, and depth," and of "length, depth, and height."

in turning, will find *one* position as a boundary the cases like the *upper* figure, where the *left*-smaller, and all the *ddle* figure, where the s the smaller? There position, as shown in There *neither* angle is other; that is, *the two*

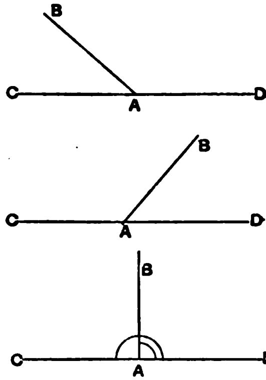


Fig. 5.

one straight line meets t line so that the two are equal, *the angles are gles*, and the lines are said *cular to each other*, or each lar to the other (23).

two or more lines make *no* ach other, as *ab*, *cd*, and *ef*, in Fig. 6, they *have the same direction*, and are said to be *parallel*. *Parallels* are everywhere at the *same distance apart*, and *never meet*.


Fig. 6.

il other angles, such as BAC or BAD in the upper figures ,, are called *oblique angles*. If *less* than a right angle, called *acute;* if *greater* than a right angle, they are *tuse*.

**actice.**—1. When one line *meets* another without crossing it, what med?

How do these angles usually compare with each other in size?

What two classes of cases thus arise?

4. In the case which is the limit, or boundary, between these classes, lo the angles compare with each other?

5. In the last case what are the angles called?

6. What *position* are the two lines then said to have with respect to other?

7. What *angle* do parallel lines make with each other?

8. How do they compare as to *direction?*

9. Give examples of right angles in buildings, rooms, furniture, fields, streets, etc.

10. Mention examples of parallel lines.

11. What are *oblique* angles?

12. When are they called *obtuse?* when *acute?*

13. Mention some examples of acute and of obtuse angles.

14. Draw a right angle as correctly as you can on paper, slate, or blackboard.

15. Draw two acute angles of different sizes.

16. Draw two obtuse angles of different sizes.

**37.** The regular, or *geometrical*, forms, besides angles, are so many that we cannot even name and describe them all. We shall only study the most common and useful ones. Even these are too numerous to mention separately here; but, fortunately, they can all be arranged in a few classes or families, so that all the varieties in each class shall be more like each other than they are like any thing else.

**38.** *When a point moves in space*, it describes a *line* of some kind. This is seen by rapidly whirling a lighted match irregularly in the dark.

When a point moves directly towards a fixed point, it describes a *straight line*.

**39.** From the spinning of a top or the whirling of a button tied to a string, we become familiar with the kind of motion called revolution. If a point revolves around a straight line, as a white dot on the rim of a wheel revolves around the axle, the point will change its direction at every instant, and will describe the line commonly called a *circle*. The fixed point in the axle, about which the wheel turns, is the *centre* of the circle. All the points of the circle are at the same distance from the centre. This distance is called the *radius*.

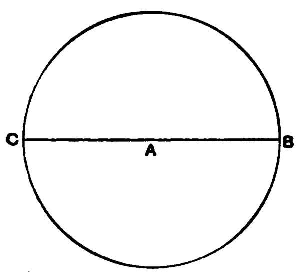

Fig. 7.

Fig. 7 shows a circle with its centre, A, and radius, AB, and *diameter*, CAB.

Sometimes the word "circle" means the entire space within the circular line. This line is then distinguished as the *circumference of the circle*. *Any* straight line from A to the circumference is a *radius*, and *any one* through A, and limited at both ends by the circumference, is a *diameter*. This last evidently equals *twice the radius*.

**40.** Whenever a point moves so as to change its direction at every instant, — that is, continually, as in the path of a stone, or of a bird in the air, or of a snowflake in a whirlwind, — it describes a *curve line* of some kind. The boundary of any figure (29) bounded by a curved line, as an irregular pond, is the *circumference* of that figure.

When we speak of a line without mentioning what kind of line, a straight line is to be understood.

**Practice.**—1. What is formed by the motion of a point? Illustrate this.

2. How must a point move in order to make a straight line?
3. Give examples of straight lines.
4. Give examples of the kind of motion called revolution.
5. How does a point move in order to describe a circle?
6. When "circle" means all the space within the curved boundary, what is this bounding line called?
7. What is the *centre*, the *radius*, the *diameter*, of a circle?
8. How does the diameter compare with the radius in any circle?
9. How often does a point change its direction in moving so as to form a circle?
10. If a point moves freely in space, not by revolving about a fixed axis, what does it describe?
11. Give examples of a point moving so as to describe any curve not a circle.
12. What is the boundary of any curved figure called?
13. How do the *lengths of different radii of the same circle* compare with each other?
14. Also the lengths of the diameters of the same circle?
15. How many radii and diameters has any circle?
16. Mention examples of circular forms (circles or parts of circles) in buildings, fruits, carriages, etc.
17. What *kind* of line is generally meant when we speak simply of *a line*?
18. What kind of a line would a point describe if it should move towards another point which should be constantly changing its position?

### Surfaces. Plane Surfaces.

**41.** If, instead of revolving a *point* about a fixed straight line, we *revolve a straight line* about a fixed straight line, the revolving line is said to describe, or form, or sweep over, or *generate* (that is, cause to be), a *surface*.

*The character of the surface* will depend on the *position* of the revolving line with respect to the fixed one.

**42.** The *simplest case* will be found to be that in which the revolving line is *perpendicular* to the fixed line,—or *axis*, as it is called. To illustrate this case, suppose a knitting-needle to be run through the stem of a top. Then, when the top is spinning steadily as fast as it can, the needle, spinning with it, will seem to form a little gray sheet or flat surface in the air. This will be a *plane surface.*

Thus, when a straight line revolves about another straight line perpendicular to it, the revolving line generates a *plane surface.* The longer the revolving line, the larger will be the plane surface. If there be no end to this line, the plane will also be without limit. Such a plane would divide the whole of space into two equal parts.

**43.** A plane can be generated in other ways; for example, by a straight line which moves upon another straight line, and always keeps parallel to its own first position. Thus, if a straight stick or rod be moved along, resting on the edge of a table, and keeping its successive positions parallel to each other, it will generate a plane surface.

**44.** Again, if a straight line moves upon two fixed lines which intersect (that is, meet each other) or which are parallel, it will generate a plane. Thus, lay two yard-sticks upon a table, and either parallel to each other or not; then a third yard-stick, moving upon the two fixed ones, will generate a plane.

Set two boards edgewise in a heap of sand, and so that their top edges shall meet at some point. Then, with a third board placed crosswise of these fixed boards, scrape down the sand to a level with their top edges. You will thus see a plane formed by the method just described.

**45.** Examples. — Floors are plane surfaces; so are flat ceilings, or the walls of rooms generally. A level lawn is a plane surface; so is the slope of the roof of a house.

*The straight line which joins any two points whatever of a plane, will lie wholly in that plane;* in other words, from any point in a plane, any number of straight lines can be drawn which will be wholly in the plane.

### Figures.

**46.** Any portion of any surface bounded by lines is called a *figure* (29). If it is a portion of a plane surface, it is a *plane figure.*

Examples. — A floor with its bounding edges, or a level field with its bounding fences, is a plane figure. A, B, and S, Fig. 8, are plane figures.

A is a *triangle;* that is, a three-cornered figure. S, having four equal sides, is a *square.*

The *triangle*, *square*, and *circle* (Fig. 7), are the most universally useful plane figures, and a large part of geometry is occupied with the study of these and others.

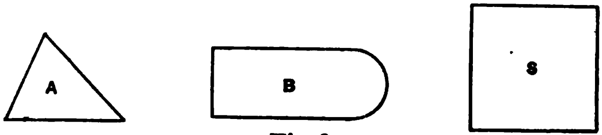


**Fig. 8.**

**Practice.** — 1. What will be generated by the revolution of a straight line about a fixed straight line?

2. What is the fixed line called?

3. When the revolving line is perpendicular to the fixed line, what will be generated?

4. Make a little machine to illustrate this by passing a cord round a grooved circular board a foot in diameter, and a spool, both turning in supports on the same stand. Attach a white or bright-colored wire rod to the axis of the spool at right angles with it, and then, by a string wound around the axis of the large wheel, or in some other way, revolve the large wheel very rapidly. The wire will then revolve fast enough to appear to form a hazy plane.

5. How can an unlimited plane surface be conceived to be thus generated?

6. Mention another way of generating a plane.

7. Mention a third way of generating a plane.

8. Give some examples of plane surfaces.

9. What is true of the straight line which joins any two points of a plane?

10. In how many directions can a line be drawn in a plane from any point in the plane?

11. What is a figure? What is a *plane* figure?

12. Which are the most universally useful plane figures?

13. What is a triangle?

14. What is a square?

### Solids. Cylinders and Prisms.

**47.** Continuing the study of one straight line revolving about a fixed straight line, we will now consider the *revolving line as parallel* to the fixed line, or axis.

Thus, let the line AB, Fig. 9, or 10, revolve about the axis CD. Every point of the revolving line will describe a circle about a point in CD as a centre; and as AB and CD are parallel, or everywhere at the same distance apart, all these circles will be equal.

The body thus formed is called a *cylinder*. The line CD, joining the centres of any two of its equal circles, is its axis. Both figures alike represent cylinders. One, Fig. 9, represents a *vertical cylinder*, because its axis, CD, is vertical (18). The other, Fig. 10, represents a *horizontal cylinder*. Thus, what a body really is in itself, and hence its name, depends, as said before (11), on its form or the manner in which it is formed.

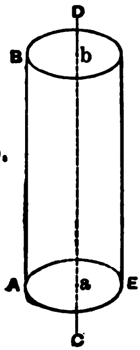


Fig. 9.

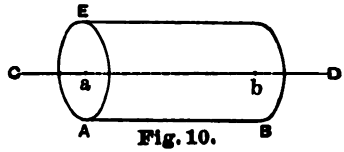


Fig. 10.

**48.** The distance, AE, through a cylinder, perpendicular to its axis, is called its *diameter*. As all the diameters are equal, the *diameter* AE and *length* AB are all the *dimensions* (25) that need be known in order to make a cylinder of a *required size*.

**49.** If any other curve than a circle should take the place of the circle AEC, and if AB were drawn to make *any* angle with AE, and if lines parallel to that position of AB were then drawn at every point of such curve, the body formed would still be called a cylinder. But we have now to do only with the cylinder formed as shown in Figs. 9 and 10, and hence called a *cylinder of revolution*. It is also called a *right cylinder with a circular base*, because the plane ends, called bases, are circles, and the curved surface formed by AB is at right angles to the bases.

**50.** Examples. — The cylinder of revolution is oftenest found in practical use. Many tin pails, most stove-pipe, round lead-pencils, the round blocks that ribbons are wound upon, round paper or tin boxes, gas and water pipes, round bottles, circular buildings, and many other objects, are wholly or mostly cylinders.

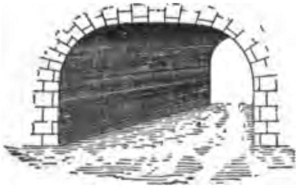

Fig. 10(a)

A brick drain or aqueduct, one of whose diameters is greater than the other, or an oval or flattened arch, as in Fig. 10*a*, are examples of cylindrical surfaces, *not* of revolution.

**51.** If we replace the equal circular bases of the cylinder by two equal plane figures bounded by straight lines, and similarly placed, the cylinder will become a *prism*. A prism accordingly differs from a cylinder only in having a base bounded by straight lines instead of by a circle. Fig. 11 represents a four-sided prism.

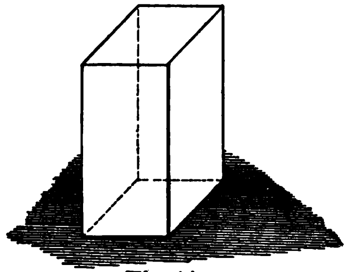

Fig. 11.

**52.** EXAMPLES. — Any room having straight or plane walls and a flat or plane ceiling is a prism. Most wooden, tin, or paper boxes having flat sides are prisms. A brick is a prism. A tile is a thin prism. Most buildings and parts of buildings, excepting the roofs, and many articles of furniture, are prisms or collections of prisms, only with ornamental bands or other additions.

**Practice.** — 1. What will be formed by revolving a line about an axis parallel to it?

2. What will each point of the revolving line describe?

3. What is the line called which joins the centres of the two ends of a cylinder?

4. When is the cylinder itself said to be vertical, horizontal, or inclined?

5. Does the *position* of the cylinder affect its *form*, *size*, or *name*?

6. What is the *diameter* of a cylinder?

7. Which dimensions are sufficient to enable one to make a cylinder of any required size?

8. Out of a wooden bar one inch and a half square, make a cylinder four inches long and one inch and a half diameter.

9. Make a paper cylinder of the same size.

10. Are there any other cylinders than those of revolution?

11. What else are cylinders of revolution called? and why?

12. What kind of cylinder is most used in the mechanic arts?

13. Did you ever see any cylinder, but one of revolution?

14. If the curved surface of a cylinder is replaced by a number of square-cornered flat surfaces, as in a four, six, or eight sided post, monument, or chimney, what is the body thus formed called?

15. How do the two ends or bases of a prism differ from those of a cylinder?

16. Mention some examples of prisms.

### Cones and Pyramids.

**53.** If the revolving line which we have all along been considering (47) meets the fixed line or axis around which it revolves at an acute angle, its motion will generate the surface of a *cone* (Fig. 12).

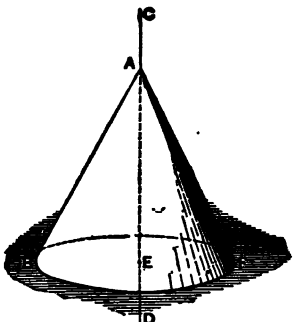

Fig. 12.

Thus, if the straight line AB revolves about the axis CD, it will generate the cone ABF. Hence straight lines can be drawn from A to every point of the circumference of the base.

The cone, like the cylinder, may be in any *position*, and may have any other curve than a circle for its base; but the cone above described is the only kind that we need to consider now.

**54.** Also, as in the case of the cylinder, if, instead of a circle for the base, we take a straight-sided plane figure, the cone will become a pyramid, as shown in Fig. 13, which represents a four-sided pyramid. The characteristic point, A, of the cone or pyramid, is called its *vertex*. The total surface between the two bases of a cylinder or a prism, or between the vertex and the base of a cone or a pyramid, is called the *convex surface*. Complete cones and pyramids are not so often found in practice as cylinders and prisms are.

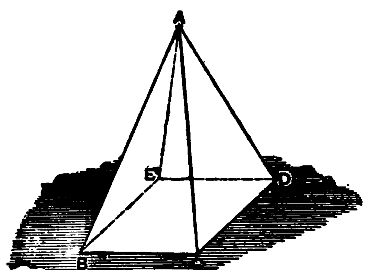

Fig. 13.

The roof of a round (cylindrical) tower may be a cone, and that of a square tower or house may be a pyramid.

Parts of cones and pyramids, not continued to the vertex, are, however, often seen in common objects. Such objects are said to be of conical or of pyramidal form.

**55.** Examples. — Tubs, pails, and firkins are conical. Oval tubs with sloping sides are examples of conical surfaces in which the base is not circular. Paper lamp-shades and the tin tunnels used in pouring liquids into bottles are conical. Tapering broomhandles are examples of very long conical surfaces, and the nearly flat tin covers of large iron pots are examples of very short or nearly flat cones.

Stone posts are often finished with pyramidal tops.

### Limits.

**56.** In studying the right angle (33, 34), we gained the idea of a *limit*. There can be many *acute* angles and many *obtuse* angles (Fig. 5), but there can be only one right angle; and we saw that the right angle was the *limit*, or boundary, *between* two opposite cases.

We shall now see that there is a long series of different *cones*, with a *limiting case at each end* of the series. If, in Fig. 12, we suppose the vertex of the cone to be farther and farther from A on the line EA prolonged, the straight lines, as BA and FA of the surface, will be more and more nearly parallel to the axis EA, until, when A reaches an *infinite* distance from E, BA, FA, etc., will be just the same as parallel to EA, and then the cone will become a *cylinder* (47).

**57.** Again, as A draws nearer and nearer to E, the cone becomes flatter and flatter, until, when A falls at E, the cone will become a *plane*, generated as described in (42).

Thus we find that there is an infinite series of cones, ending at one end of the series in the *cylinder*, and at the other end in a *plane*.

### The Sphere and Torus.

**58.** We now leave those surfaces which can be formed by the motion of a straight line, and on which, therefore, it is possible to draw straight lines in certain directions, and come to those which can only be formed by the motion of a curved line, and on which, therefore, no straight line can be drawn.

We shall here consider only those curved surfaces which can be formed by the revolution of a circle about an axis.

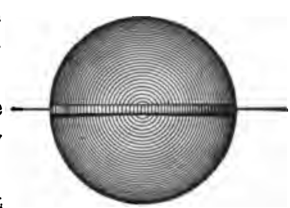

Fig. 14.

**59.** Take a quarter-dollar, a two-cent piece, or, best of all, if you can get one, an old-fashion copper cent. Take it up *by the edge* between two opposite pin-points, as in Fig. 14, and then, by blowing upon it, till it spins so fast as to appear like a ball, you will understand what is meant by saying that a sphere (Fig. 14) is generated by the revolution of a circle about any one of its diameters.

**60.** From the manner in which the sphere is formed, it is evident that it has a centre equally distant from every point of its surface, and that this centre is the same as that of the generating circle. Hence, also, the diameters of the sphere are equal to those of the same circle, and to each other, in whatever direction they are drawn.

**61.** EXAMPLES. — Spheres are seen in balls, domes, globes, and pretty nearly in some kinds of fruits and berries, as currants and oranges. The earth is very nearly a sphere, and so are most of the heavenly bodies.

A half-sphere is called a hemisphere. Round domes are often hemispherical.

**62.** Finally, let A and B, Fig. 15, be two equal thin plates, disks, or vanes, fastened to a slender rod, AB, which is made to spin with great velocity on the upright piece, CD, at C, the middle point of AB. If the vanes are of any light, bright color, they will, when revolving very rapidly, appear to form a solid ring in a horizontal position. This ring is the form called an *annular torus*. Every section of it, made by cutting it vertically, will be a circle equal to the generating circle A or B.

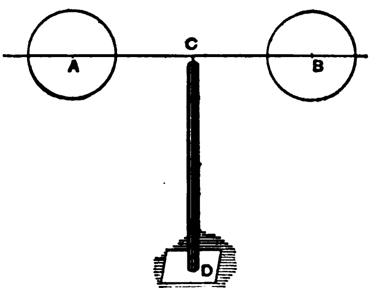

Fig. 15.

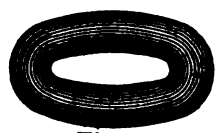
Fig. 16.

Every horizontal cutting of the torus will also show circular outlines. Fig. 16 may give an idea of this body. Examples of it are seen in all rings of round material, as when a piece of lead pipe is coiled up into a ring, in the iron rings of harnesses and yokes for horses and oxen, or in the handles of hatchways, the brass rings used as drawer-handles for furniture, etc.

**Summary.**

**63.** The motion of a *point* generates a *line*. The motion of a *line* generates a *surface*, — a surface as a cone, on which a straight line can be drawn, if the moving line be *straight;* but a surface as a sphere, on which no straight line can be drawn, if the moving line be a *curve* all of whose points move in curves. Finally, the motion of a *surface* (except that of a plane figure not leaving its own plane) generates a *solid*, as when a hot slate is dropped into a snow-bank.

**64.** We now see that solids can be divided, in geometry (37), into *two great classes*, — those which can be formed by the motion of a straight line, as cylinders, etc.; and those which cannot be so formed, as the sphere and others. Also, that those which *can* be formed by the motion of a straight line can be arranged in two divisions, — pyramids and cones, which *have a vertex;* and prisms and cylinders, which *have not* one.

**Practice.** — 1. How is a cone generated?

2. What is it called if a straight-sided figure takes the place of the circular base?

3. What is the point of a cone or pyramid called?

4. What the curved or flat-sided surface?

5. Mention examples of complete or of incomplete conical surfaces.

6. Also of pyramidal surfaces.

7. What does the cone finally become when its axis is lengthened without limit?

8. What when it is shortened to nothing?

9. What, then, are the two extreme or limiting forms of the cone?

10. How many forms between these limits?

11. What kind of line can be drawn in one or more directions on the plane, cylinder, and cone?

12. What other class of surfaces is there?

13. Illustrate the generation of a sphere by a circle.

14. Define a sphere by the relation of its surface to its centre.

15. How many equal diameters has a sphere?

16. Mention examples of spherical surfaces.

17. Describe the generation and form of an annular torus.

18. Give an example of the generation of a solid by a surface.

# CHAPTER II.

## MEASURING AND DRAWING.

### Instruments and Operations.

**65.** Things are so much better understood by *seeing* them than by only hearing or reading about them, that it is a great advantage, in the study of geometry, to *draw the figures* that are studied. Besides, we thus not only see them, but *make them ourselves*, and so become much better acquainted with them.

But, in order to draw an object correctly, we must know the object itself with respect to its *size*, *form*, and *position*. We must also possess some means of representing it as we wish.

**66.** We find the size and position of the object to be drawn by measuring it, and its distance from us or from other objects. But, as most objects are too large to be drawn as large as they really are, we represent them by smaller figures, made in such a way as will call to mind the objects themselves.

Before proceeding to measure and draw objects, we must learn something of measuring and drawing instruments. This we will next do.

**67.** *Instruments and Materials.* — Drawings which are made by the aid of instruments are more conveniently made on separate sheets or plates than in a drawing-book.

All such figures or drawings as are needed in connection with the study of this book can conveniently be made on half-sheets of unruled letter-paper, or other white or buff paper of about that size.

Then a moderately hard lead-pencil, a piece of india-rubber, and a ruler about eight inches long, will be necessary, and an atlas or portfolio, or something equivalent, to lay the paper on while drawing.

68. Very cheap sets of the necessary instruments for school purposes can now be bought; but nothing but a pair of compasses (Fig. 30), with one leg movable, to be replaced by a pencil-point, is indispensable. The learner can easily find means for marking inches, with their halves and quarters and eighths, on the edge of a common ruler, or of a slip of stout cardboard, such as can be cut, if need be, from a paper box. From the same material can also be made the triangles of the form used in making all such figures, as Figs. 25 and 26.

69. The triangle used in drawing the lines, as AC of Fig. 25, and others of the same kind, are of the form of ABC, half of a square, ABCD, as shown in Fig. 17.

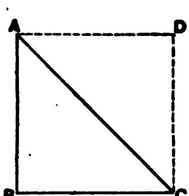

Fig. 17.

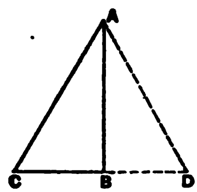

Fig. 18

That which is used to draw the lines, as *ab* and *ac* in Fig. 26, and all figures of that kind, is of the form ABC in Fig. 18, or half of a triangle ACD, *all* of whose sides are equal.

Each of these triangles thus contains a *right angle*, B.

70. Pencils can be sharpened very nicely on a fine file or bit of sandpaper, and with less danger of breaking the point than when a knife is used.

71. A vertical (18) line may be drawn on the blackboard, but not on paper lying on a desk or table. Yet, in pictures of buildings, we call the lines which are perpendicular to the top or bottom of the sheet, vertical lines, because they *represent* lines which *are* vertical on the object itself. Also, when the picture hangs on the wall, these lines become truly vertical. Hence, when we speak of lines on the paper as vertical lines, we mean those which represent vertical lines, or are perpendicular to the bottom edge of the sheet, or the one nearest the person.

72. To assist the eye in estimating the two principal directions correctly, a border line may be ruled, not over one inch from the edges of the paper. This also gives the plates a neater appearance, and a margin for handling or binding.

**73.** The two most frequent relative directions in which lines have to be drawn are *parallel to each other*, and *perpendicular to each other.* We will show how to draw them thus by means of the triangles already described.

*Parallels.* — Let it be required to draw a parallel to the line AB, through any point, as *p*, Fig. 19.

1°. Place *any* edge, as AC, of *either* triangle, against the *given line* AB.

2°. Bring up a ruler, AD, or the other triangle, for a *fixed support* against the edge AD of the triangle ACD.

3°. Holding the ruler AD (or the other triangle) firmly against the paper, to prevent its slipping, slide the triangle ACD on the ruler, till, as at *acd*, the edge *ac*, which was on AB, passes through *p*.

4°. Then, holding the position *acd* firmly, draw *ac*, which will be parallel to AB. Any number of parallels to AB can be drawn in like manner.

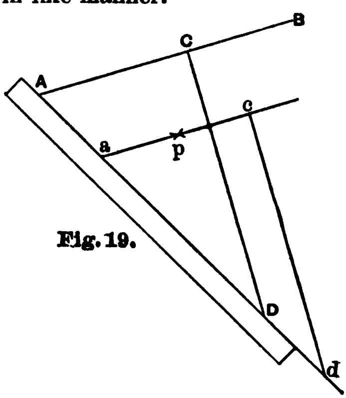

Fig. 19.

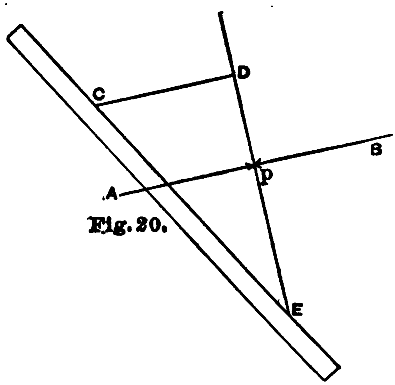

Fig. 20.

**74.** *Perpendiculars* (Fig. 20). — Let it be required to draw a perpendicular to AB at *p*.

1°. Place *either of the sides*, as CD, *of the right angle of either triangle* against the line AB, and then bring up the ruler (or the other triangle) for a fixed support against the *longest edge*, CE, of the triangle.

2°. Then slide the triangle on the ruler *until the other side*, DE, of the *right angle* passes through the given point *p*, whether that point be on or off the given line AB, and then draw the required perpendicular, DE, to AB.

**75.** After the definitions of points and lines already given, it will be understood that what are called lines, in a figure drawn on paper or a blackboard, are really very narrow bands of pencil-lead, ink, color, or chalk, which only represent lines to the mind. So a point on paper, etc., is a very small surface of pencil-lead, ink, or chalk, which marks to the eye the place where the true point which the mind is thinking of is meant to be.

**76.** When it is desired to make a drawing as exact as possible, as in the drawing of fine machinery, the lines are made as fine as possible with a very hard pencil, sharpened to a thin edge instead of to a round point. The points through which the lines are to pass are marked by faintly pricking the paper with a fine needle-point, and then making a little circle round each point, or by two very fine marks through it, to catch the eye. This *extreme niceness* will not generally be necessary, however, in the study of this book, or in most carpenter's, tinsmith's, and other kinds of mechanical work.

*Distances* and *directions* should be accurately laid down. Every operation, indeed, should be as true as the means allow, or as the case properly requires.

**Practice.**—1. What must we know and have in order to make the drawing of an object?

2. How does the drawing generally compare with the object in size?

3. What materials are necessary for drawing?

4. Make a pair of triangles, of the forms described, of cardboard, tin, zinc, thin wood, or other convenient material.

5. Mark the longest side of each, in inches, halves, quarters, and eighths, or in inches and tenths, as accurately as possible.

6. Draw parallels with the ruler alone, placing it by the eye.

7. Draw parallels with the ruler and triangle.

8. Draw perpendiculars in various positions by the eye, using the ruler to draw the lines.

9. Test them by the ruler and triangle.

10. Draw parallels with the *longest* side of the triangle, either side of the *right angle* resting against the ruler.

11. Find by trial how best to handle the ruler and triangle in drawing *perpendiculars* when the longest side of the triangle is placed against AB (Fig. 20).

12. Draw three perpendiculars to a nearly horizontal given line, one through a point *on* the line, one through a point *above* it, one through a point *below* it.

13. Do the like to a line nearly vertical, the points being on it, and to the right and left of it.

**Systems of Measures.**

**77.** In finding for itself a system of measures, the human mind seems to have moved in different ways at different times and places.

Man, himself having two hands and two eyes, and being surrounded by hosts of three or five parted flowers in the plant-world, the *English* mind seems to have felt its way to a system based on the numbers 2, 3, and 5, and thus giving *halves*, *thirds*, and *fifths* of units; and, based upon these, *fourths*, *sixths*, *eighths*, *tenths*, *twelfths*, etc., of units, all of which have been shown by the experience of ages to meet the familiar wants of common life.

**78.** But, again, the acute *French* mind, noting the great convenience in computation of the decimal system of numbers, and stirred to the utmost activity in every direction at the time of that mighty upheaval of every thing, the French Revolution of 1789, then conceived the idea of a system of measures which should be wholly decimal, and thence produced the metric system, as it is called, from the meter, which is its unit of measure. The meter is then divided into tenths, hundredths, etc., or multiplied by ten, one hundred, etc.

**79.** Here, then, we have the two rival candidates for the world's favor, — the English system, with its adaptation, under the guidance of long experience and natural suggestion, to the felt wants of all the varied forms of common life, with respect to convenient *sizes* and fractional *divisions* of wholes; and the French system, with its evidently superior convenience in computation, whether or not equally good in other respects.

We shall exhibit both, and give examples in the use of both, with occasional hints in aid of a careful comparison between them.

**80.** First, the English *foot*, which is the same as that of the United States, is derived from the length of a pendulum beating seconds at London.

Second, the French *meter* is the one ten-millionth part of the distance from the equator to one of the poles of the earth, and is equal to 3.2809 English feet, or 39.3708 English inches.

Thence we further have (79), of *parts of the meter*, —

| | | | | | |
|---|---|---|---|---|---|
| 1 decimeter (dm.) | = 0.1 | meter (m.) | = 3.94 | inches | = nearly 4 inches. |
| 1 centimeter (cm.) | = 0.01 | " | = 0.394 | inch | = nearly $\frac{2}{5}$ inch. |
| 1 millimeter (mm.) | = 0.001 | " | = 0.0394 | inch | = nearly $\frac{1}{25}$ inch. |

Also, of *multiples of the meter*, —

| | | | | |
|---|---|---|---|---|
| 1 dekameter (dkm.) | = 10 meters | = 10.936 yards | = nearly | 2 rods. |
| 1 hektometer (hm.) | = 100 " | = 109.36 yards | = nearly | 20 rods. |
| 1 kilometer (km.) | = 1,000 " | = 1,093.6 yards | = nearly | $\frac{5}{8}$ miles. |

**81.** The parentheses contain the abbreviations of the various denominations.

The contrast is noticeable between the length of these names and that of the familiar monosyllables, inch, foot, yard, rod, and mile

Further, it is a natural thought that the three former, — the inch, foot, and yard, — agreeing closely with the measures of the average human finger-joint, foot, and arm (from the spine to the tip of the outspread arm), are naturally adapted to measure such things as man most uses, and which are therefore adapted to his hands, and thence called "handy," such as doors, windows, books, furniture, vehicles, etc.

**Practice.** — 1. On what numbers are the English measures founded?

2. What numerous different fractional divisions do they therefore afford?

3. What may have led to the decimal system of measures?

4. When and by whom was it devised?

5. What is its marked point of superiority?

6. What two points in the English system have resulted from long experience?

7. From what are the English foot and the French meter derived?

8. Construct a table showing the value of *each* metric unit in *all* of the lower units, 1 km. = 10 hm. = etc.

9. 75 meters are how many decimeters? How many dekameters?

10. Reduce one *mile* to *feet*.

11. Reduce one *kilometer* to decimeters.

12. 40 kilometers an hour is how many *miles* an hour?

13. 40 miles an hour is how many *kilometers* an hour?

14. Make the measures, and find, by repeated trial, whether you would prefer to measure a door, window, table, bed, fireplace, box, fence-post, with a meter-stick (39.3 inches long), a yardstick (36 inches), a decimeter-rule (not quite 4 inches), or a 2-foot rule.

15. If you were laying out a garden, find likewise which you would prefer, — a meter-stick, a dekameter-pole (about 33 feet long), a rod-pole ($16\frac{1}{2}$ feet), or a 10-foot pole?

16. What *fractions* of the dekameter-pole (which would be heavy, and hitting every thing) would be nearest in length to the rod and the 10-foot pole?

17. But if you had a tape-measure one *dekameter* long, and divided into *meters*, *decimeters*, and *centimeters*, would you prefer to know the measure of your house in *any two or more* of these measures, or in feet and inches?

18. I am writing on paper 7¾ inches by 9¾ inches. This is very nearly 2 decimeters by 2½ decimeters (very simple numbers), or 19.7 centimeters by 24.8 centimeters. Having regard to the very infrequent use of the *decimeter* (about the length of the forefinger), and the small size of the *centimeter* (giving many of them in a short distance), find by trial whether you would prefer either of them to the inch for measuring all sorts of *handy* articles (that is, articles suited to the *hand*).

## Drawings.

**82.** We will now proceed to explain how various useful kinds of drawings are made by the aid of the instruments and measures which we have now described.

**First,** *Drawings which show the Forms of Objects as they really are.* — A brick is *eight* inches long, *four* inches wide, and *two* inches thick. See now these three parts of one figure (Fig. 21). A is made *eight-sixths* of an inch long and *four-sixths* of an inch wide. It therefore represents the true *form* of the largest surface or *top* of a brick, its *length* and *width*. B is *eight-sixths* of an inch long and *two-sixths* high, and therefore represents the real form of the *front* of a brick, or its length and thickness, as seen in that face of a brick which we should see in the face of a brick wall. C is *four-sixths* of an inch wide and *two-sixths* of an inch high, and therefore shows the true *form* of the *end* of a brick, as we often see it at the corners of a brick house.

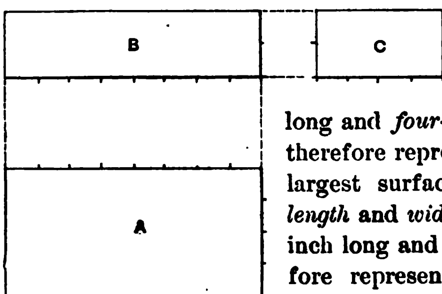

Fig. 21.

**83.** The three parts, A, B, and C, of the figure, or, indeed, *any two* of them, show to us the real form of a brick, and its *three dimensions*, — *length*, *breadth*, and *thickness*, — represented or expressed, in *sixths* of an inch.

A is called the *plan* of the brick, B is called the *front elevation* of the brick, and C is called the *end elevation* of the brick. As every *inch on the real brick* is represented on the drawing by *one-sixth of an inch*, we say that Fig. 21 is *a brick drawn on a scale of one-sixth of an inch to one inch*, or a *scale of one-sixth.*

**84.** This, to be sure, is a very simple case; but it will show what architects, builders, machinists, surveyors, and others, mean when they are speaking of *plans and elevations* of buildings and other structures, and of "*drawing to scale*," or the "scale of a drawing."

Drawings of this kind are called *projections*. They are also called *working drawings*, because the objects represented can be made just as they are meant to be, from the information given by the drawing as to their *size* and *form*.

**85. Second,** *Drawings which represent Objects as they appear.* — By placing the eye about three inches in front of the corner of the fence in Fig. 22 (with one eye closed), it will be seen that the figure very well represents a garden containing trees and shrubs at the corner of a street, and enclosed by a board fence.

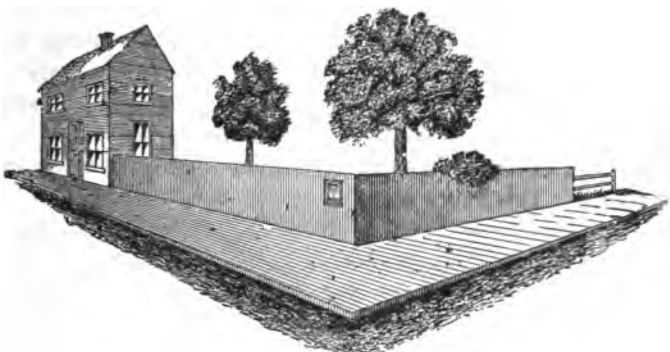

Fig. 22.

This figure is of the kind called a *picture*, or a *perspective drawing*. Such drawing does not represent objects as they *really are in themselves*, as Fig. 21 does, but as they *appear* to the eye according to their different distances from it. Objects which are *really* of the *same size*, but at different distances from us, *appear* smaller the farther away they are from us. Perspective drawing takes account of this fact. It is sometimes used in geometry in representing solid bodies. Photographs of real objects; also portraits, landscape and flower paintings, and artists' drawings, generally, are perspectives.

**86. Third,** *Drawings having Pictorial Effect, yet showing Real Dimensions.* — In representing small objects, like boxes, blocks, etc., *our knowledge of their real size and form*, combined with the fact, that, when they are a few feet from the eye, *all their parts are not far from the same distance away*, makes the kind of drawings shown in Figs. 23 and 24 quite agreeable, and easy to understand. Nearly all the figures of solids in this book are made in these ways. On account of their character, both kinds are sometimes called *mechanical perspectives*, though in reality they are a particular kind of projections, or working drawings (84).

**87.** Fig. 23, called an *oblique projection*, represents a brick on the same scale, of one-sixth of an inch to an inch, that was used in Fig. 21; and it certainly reminds one of a brick more readily than Fig. 21 does, though only the *front*, B, shows the true *form*, as well as the size, according to the scale. A and C are misshapen, though drawn so as to show the length and width of the brick truly by the scale.

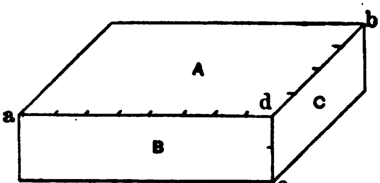


Fig. 23.

**88.** Fig. 24 is a simple example of the kind of drawing called *isometrical*. This figure seems to represent the proportions of a brick better than Fig. 23 does, though none of the faces are shown in their real *form*. Though both figures are drawn on the same scale of *one-sixth*, Fig. 23 appears to deceive the eye by making the object look shorter and wider than a brick does.

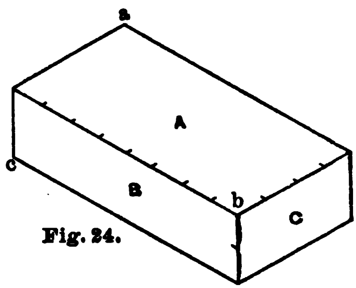


Fig. 24.

**89.** In both figures alike, 23 and 24, it makes no difference which end of the brick, or any other object, is shown. In Fig. 24, by making the lines of *width* incline to the *left*, instead of, as now, to the right, the left end of the brick would have been shown. In Fig. 23 the same result would have been obtained by making the *longer* lines incline to the right instead of the shorter ones, or by making *bd* and the parallels to it incline to the *left*.

**90.** In both of these kinds of drawing, it is to be remembered that on *all the distorted faces*, or on all but B, Fig. 23, *no* lines show their true size, except those shown on the figure, and parallels to them. For example, *ab* and *bc*, on either figure, do not appear in their true size according to the scale.

**91.** Figs. 25 and 26 are drawings, by both these methods, of the same object, — a stone trough 6 feet long, 3 feet wide, 2 feet high outside, and 1½ feet deep. On both of these figures the thickness of the sides of the trough, six inches, must be laid down on lines parallel to *ab* and *ac*. The scale is *one-fifth of an inch to one foot;* that is, one-fifth of an inch to twelve inches, or one-fifth of an inch to sixty-fifths of an inch; that is, it is a scale of *one-sixtieth* of the real size of the trough. Fig. 25 has the slight advantage of showing the front of the trough in its real *form* as well as *dimensions;* and Fig. 26, that of showing the whole depth and part of the bottom of the trough.

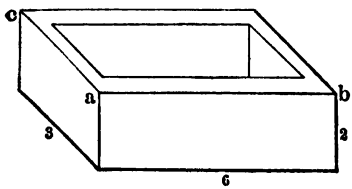

Fig. 25.

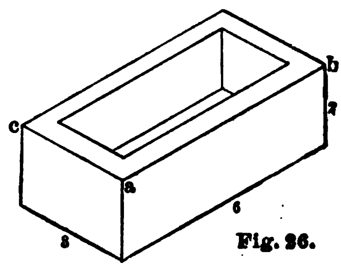

Fig. 26.

These illustrations and directions alone are a sufficient guide — along with a knowledge of the necessary instruments — to the drawing of any object having only square corners, by either of the two methods just explained. They are most useful as applied to such objects, as will be seen from many of the figures in this volume.

**Practice.** — 1. Explain how to draw a brick so as to show all its dimensions, and its top, front, and one end.

2. Make the drawing *one-fourth* of the real size of the brick.

3. Make a drawing, by the method of Fig. 23, of a brick standing on one end on a table, and you facing its broadest face.

4. Also when in the same position, but as if you were facing its narrower face.

5. Draw the brick by the same method, supposing its 8 by 2 inch face to be resting on the table.

6. Draw the brick as if its largest face rested on the table, but as if you faced one end of the brick.

7. Explain the difference between a working drawing and a picture.

8. Make, as well as you can, a picture drawing of any one or more of the following; viz., a table, chair, sofa, bureau, bookcase, wagon, monument, short flight of steps, porch, house, corner of a garden, view out of the window as you stand in the middle of the room.

9. Explain the kind of drawing called *oblique projection.*

10. Explain *isometrical drawing.*

11. What is the advantage of both of the last two kinds, over common projections for small regular objects?

12. In both kinds, which lines show their real dimensions?

13. Why are not these kinds of drawings usefully applied except to square-cornered bodies?

14. Make an *oblique projection* of a work-box, or of a work-table, or writing-table drawer with its partitions.

15. Make an *isometrical drawing* of a square stone post, a flight of four plain stone steps, or a partitioned box, tray, or drawer.

### Angles and Circular Measure.

**92.** The lines which meet to form an angle are the *sides* of the angle. They are commonly straight.

It is the *difference of direction*, and not the *length* of the sides, which makes the size of an angle.

**93.** The point where the sides of the angle meet is called the *vertex* of the angle. In speaking of two or more angles, these points are called their *vertices.*

When a number of angles have the same vertex, as is the case when a number of lines are drawn through the same point, it is called their *common vertex.*

**94.** An angle may be named in three ways, all shown in Fig. 27: —

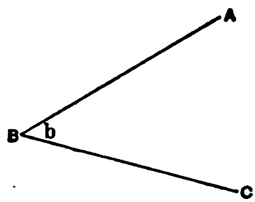

Fig. 27.

*First*, by a small letter within the angle, as *b*.

*Second*, by a capital letter at the vertex, as at B.

*Third*, by three letters, capital or small, of which the middle one marks the vertex.

Either of the first two methods is best when the figure is so simple that there can be no doubt as to what the letter means.

The third way is used to make it perfectly unmistakable what angle is meant, as we shall see in future pages.

**Practice.** — 1. What are the sides of an angle?

2. Draw a small angle with long sides, and a large angle with short sides.

3. Draw a small angle with short sides, and a large angle with long sides.

4. What is the vertex of an angle? and what are those of a number of separate angles called?

**5. When do angles have a common vertex?**

**6. Draw three separate angles, and letter no two of them the same way.**

**7. Which way cannot well be used for very small angles?**

**8. Which way cannot be used when several angles have a common vertex?**

**95.** Since an angle is the difference in direction between two lines, we may conceive of an angle as formed by a line which turns about one of its points so as to take a new direction. Thus, if the line OA turns about the point O, Fig. 28, and without leaving the paper, it will form, or, as is also said, describe, the angle AOB.

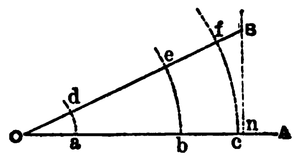


Fig. 28.

In describing this angle, every point of the line, except O, moves in a circular path, as *a*, *b*, *c*, describe the paths *ad*, *be*, *cf*.

**96.** The difference in direction of the sides of an angle is not expressed by the length — expressed in inches or other linear measure — of any line, straight or curved, as B*n* or *cf* between the sides of the angle. It is expressed by the fractional part which the circular path, as *be*, of any point of the moving line is of the entire circle described by the same point, *b*, with radius, as O*b*.

**97.** *Degrees, etc.* — Every circle is, for convenience, supposed to be divided into *three hundred and sixty* equal parts, called *degrees*. One degree is, therefore, one three hundred and sixtieth *part* of a circle. Hence it is plain, that, as a degree is not a thing of fixed length, like a foot or a yard, but is only a *fractional part* of a circle, one degree upon a large circle will be larger than one degree on a small circle.

**98.** A little circle, thus (°), is the sign of degrees; so that, for example, forty-five degrees is written thus: 45°. Also, each degree is divided into sixty equal parts, called minutes, marked thus (′); and each minute is divided into sixty equal parts, called seconds, and marked thus (″): so that 81° 17′ 32″ is read eighty-one degrees, seventeen minutes, thirty-two seconds.

**99.** Suppose a circle to be divided into 360 degrees, and suppose radii to be drawn from the points of division to the centre: it is plain that 360 small and equal angles would be formed, all having the centre of the circle for their common vertex. Hence each of these angles is very naturally called an angle of one degree.

One of these *angles* corresponds, as we see, to an *arc* of one degree on the circumference; two of them contain, between their sides, two degrees on the circumference, etc. Therefore we conclude that the number of degrees in any angle at the centre of a circle is measured by the number of degrees in the arc of the circle included between the radii, which are the sides of the angle.

**100.** Fig. 29 illustrates more perfectly the correspondence between angles and the arcs included between their sides when the centres of the arcs are at the vertices of the angles; also how to

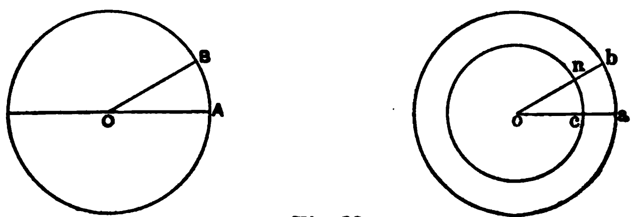


Fig. 29.

*make an angle*, as *aob*, *equal to a given angle*, as AOB. Circles are equal when their radii are equal. Then, if $oa$ = OA, the circles described with those radii are equal. Also, if *oa* is parallel to OA, and *ob* to OB, the difference of direction between *oa* and *ob* is equal to that between OA and OB, which makes the angle *aob* = the angle AOB. Then, as the circles are also equal, *ab* is the same fractional part of the circle on centre *o* that AB is of the circle on centre O; that is, arc *ab* = arc AB. Thus the equal arcs *ab* and AB measure the equal angles *aob* and AOB.

**101.** *Any* circle having the vertex of the angle for its centre can be taken, because, each side of an angle being straight, the *difference in direction* between the two sides is everywhere the same, so that *nc* is the same part of the circle of radius *oc* that *ab* is of the circle of radius *oa;* that is, *nc* has the same *number* of degrees that *ab* has, though, of course, they are of smaller size.

**102.** Whenever a figure shows what circle is meant merely by naming its centre or its radius, it is sufficient to name it thus. In Fig. 29 we may say the circle O, or the circle *oc*, or the circle *oa*.

A half-circle is called a *semicircle*.

A quarter-circle is called a *quadrant*.

**Practice.** — 1. Describe the formation of an angle by means of a revolving line.

2. What is the path of each moving point of the moving line?

3. Which point does not move?

4. Illustrate the formation of an angle by means of a jointed rule (as a folding foot or two-foot rule) or a pair of dividers (Fig. 30).

5. What does the size of an angle *not* depend on?

6. What *does* it depend on?

7. If one pair of lines start from the same point, and at three feet from that point are one foot apart, and if a second pair are one foot apart at ten feet from the meeting-point, which pair includes the larger angle?

8. How is the size of an angle expressed?

9. What is the number and the name of the parts into which every circle is supposed to be divided?

10. Are degrees of the same length for all circles, as inches are for all lines?

11. Are all the degrees on the same circle equal to each other?

12. What fractional part of 360° is 30°?

13. If 30° is a certain fractional part of a circle of one foot radius, is it the *same* fractional part of any other circle?

14. If from both ends of any arc of a circle, as one of 42°, you draw radii to the centre of the circle, what will the angle between these radii be measured by, and what will it be called?

15. A quadrant is an arc of how many degrees?

16. A right angle is an angle of how many degrees?

17. Is an arc divided into degrees by dividing the angle which includes the arc, or is the angle divided by dividing the arc included between its sides?

18. Which, then, measures the other, — the angle the arc, or the arc the angle?

**103.** For the purpose of drawing circles, an instrument (Fig. 30) called a pair of *dividers*, or compasses, is used, by removing the leg, which is held by the screw S, and replacing it with a leg formed so as to carry a bit of pencil.

Fig. 30.

**104.** For measuring the number of degrees in any portion or *arc* of a circle, the instrument called a *protractor*, and shown in Fig. 31, is used. It is made of brass, German silver, paper, or horn. By placing the instrument, with its centre, O, on O (Fig. 29), and the edge OA on the line OA (made longer if necessary), we can read off on the circular edge the number of degrees between OA and OB, and thus measure the angle.

**105.** The number 360 seems to have been chosen on account of its having a larger number of exact divisors than any other conveniently large number. The smallest whole divisor, except 1, that any number can have, is 2. The largest divisor it can have, except itself, is the one that is contained two times, or half the number. Accordingly, the divisors of 360 are the *twenty-two* following: —

2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180.

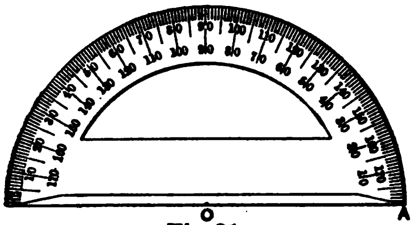
Fig. 31.

Taking the right angle as the standard angle of angular measure, the metric system would naturally give 100° to the right angle, or 400° to the circle. But the divisors of 400 are only *thirteen*, as follows: —

2, 4, 5, 8, 10, 16, 20, 25, 40, 50, 80, 100, 200.

Hence the metric system has naturally failed of adoption for circular measure.

**Practice.** — 1. With what instrument are circles drawn?

2. Open the compasses till the points are one inch apart, and draw a circle. What will be its radius? [Hold the compasses between the thumb and first two fingers of the right hand, and bear the weight of the hand a little harder on the point which is placed at the centre of the circle.]

3. Draw two diameters perpendicular to each other with the ruler and triangle.

4. Draw two lines making any angle with each other, and measure the angle with a protractor. [Or, if you do not have one, with as good a paper protractor, of about 4½ inches in diameter, as you can make by the directions before given.]

5. Draw any line, and mark a point upon it. Then at that point make an angle of 42° by the aid of the protractor. [Place the centre O (Fig. 31) at the given point.]

6. How many degrees between North and West? between West and North-west? between North-west and East?

7. If there are eighteen spokes to a wheel, any one of them makes what angle with the next one? with the third one from it?

8. Five paths make equal angles with each other at the point where they all meet. How many degrees in each angle?

9. I can walk to school in 12 minutes. That is what part of 1 hour? How many degrees does the *minute*-hand of my watch pass over in that time? the hour-hand?

10. In what time does the second-hand sweep through an angle of 60°? of 90°? of 120°? of 150°?

11. How many minute spaces are there on the circumference of a clock or watch face? How many degrees in each?

12. How many degrees in 5 minutes on the clock face? in 20 minutes? in 35 minutes?

13. One place is directly north of us, another one is one-third of the way around to the north-east of us: how many degrees in the angle between the lines from us to the two places?

14. The earth turns around on its axis once in 24 hours. Then how many degrees does it turn in 1 hour? in 3 hours? in 6 hours?

15. What angle do the hands of a clock include at 8 o'clock? at 9 o'clock? at 10 o'clock? at 12 minutes past 2 o'clock? at 24 minutes past 4 o'clock.

16. At what hour do the clock hands include no angle? one right angle? two right angles?

17. If a round cake is divided equally among 20 children, how many degrees does each get?

**106.** In every enterprise which one or more persons undertake, they first *look over the ground;* that is, they consider what things are to be thought of, and get a clear, general idea of what they are going to do. We have now done the same thing for geometry. We have looked over some of the uses of Geometry,—the ideas of Size, Form, and Position; Space; Direction and Extension; Dimensions; Magnitudes of the different *kinds*, and the different *forms* of magnitudes, and *how to represent* them.

With the general ideas thus gained, we are prepared to proceed with the more particular study of *various forms* of all the *different kinds* of magnitudes.

# CHAPTER III.

## STRAIGHT LINES.

### Definitions.

**107.** *A given point* is one which is set down at pleasure to aid in finding some desired point or line.

*A given line or figure* is one which is drawn in beginning some diagram in which some points or other lines are to be found, or about which something is to be proved.

**108.** *An indefinite line* is one of no particular given or required length.

*A definite line* is one of a certain length, marked by its intersection with other lines, or, if standing alone, by a short mark across it at each end, as the line AB, Fig. 32.

Fig. 32.

**109.** *To take up or off a distance* means to set one point of the compasses at one end of that distance, as at A (Fig. 32), and to open the compasses till the other point falls at the other end of that distance, as at B.

**110.** *To set off or lay down a distance* means to mark upon any line, with the points of the compasses, a distance taken up from some other line, as just described; or from a divided ruler or scale.

**111.** *A proposition* is something set before the mind to be thought about or acted upon. When it is something to be *proved* by a course of reasoning, it is a *theorem;* when it is something to be *done* or *found* by a series of operations, it is a *problem.*

The course of reasoning is called the *demonstration of the theorem.* The series of operations are the *solution of the problem.*

**112. Theorem.**—*A straight line is the shortest distance between two points.*

Let A and B, Fig. 33, be the two points. AB is the straight line which joins them. AB has therefore one direction. But, in passing from A to B by the path ACD, we have, *besides* going over the distance AB in the one direction AB, departed from AB by the distance DC. Now, DC being perpendicular to AB, all progress in that direction has afforded no progress in the direction AB. Hence ACD, or any like path, is longer than the one having the single direction AB.

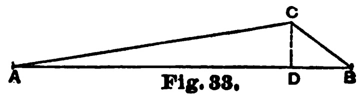

Fig. 33.

Hence the straight line AB is the shortest distance between the two points A and B.

**113.** Every boy who knows what it means to "cut across lots" in order to shorten the distance to the point where he is going, is just as well satisfied of the truth of the proposition just explained before proving it as afterward. But it is here given to show, that, if any one doubts it, something, though only a little, can be said to prove it true.

If *nothing* can be said to make a truth any plainer than it is of itself, it is said to be *self-evident*, and is called an *axiom*. The theorem just given is also often called an axiom. The following are some other axioms: —

1°. The whole equals the sum of all its parts.

2°. The whole is greater than any of its parts.

3°. Things which are equal to the same thing are equal to each other.

4°. If A equals B, and *a* equals *b*, then the sum of A and *a* is equal to the sum of B and *b*. This is what is meant by saying, as is commonly done, that, if equals be added to equals, the sums will be equal.

5°. If equals, instead of being *added* to equals, are (*a*) subtracted from them, or (*b*) multiplied by them, or (*c*) divided by them, the results will still be equals.

**Practice.** — 1. What is meant by a given point?

2. What by a given line?

3. What is an indefinite line?

4. What is a definite line? Is a line two inches long a definite, or an indefinite, line?

5. What is it to take up and to set down a distance?

6. Take up two inches from a foot rule or other scale, and set it down three times in succession on a line. Was the line definite, or indefinite, before this was done?

7. What is a proposition? a theorem? a problem?

8. What is the shortest distance between two points? Prove it.

9. How many straight lines can be drawn from one point to another?

10. If a truth is so plain that nothing can make it any plainer, what is it called?

11. If two boys have the same number of marbles, and you give each of them three more, how do you know that they will still have equal numbers of marbles?

12. If you have two equal sticks, and cut off two inches from each, how do you know that the parts that are left will be equal?

13. Place *one corner* of one-half sheet of writing-paper *at the centre* of the other half of the same sheet. Cut out the quarter-page which is thus of two thicknesses. How do you know that the rest of the half-sheets are equal?

**114.** When several successive lines of any length and direction are drawn so that each meets the next, they form a broken line, as ABCDEFG, Fig. 34.

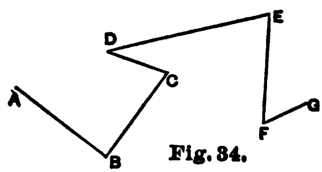


Fig. 34.

**115.** When the parts of a broken line are arranged in any regular manner, they become ornamental, as seen in Fig. 35, which may represent a band of triangular pieces in the border of a floor, or of small raised pieces on the border of a piece of furniture.

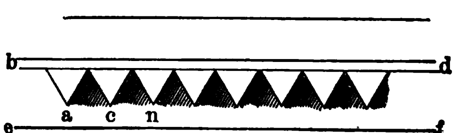


Fig. 35.

In the former case contrast of color would make them beautiful; in the latter case the contrast of light and shade between the pieces and the spaces between them would have the same effect.

**Practice.** — 1. Experiment with other angles than that of 60°, shown in Fig. 35, as 45° or 90°, and judge of the effect as better or worse.

2. Experiment on unequal angles at *a* and *c*; that is, make only alternate angles, as *a* and *n*, equal, and note the result.

3. Experiment with *unequal* angles between the sides of the points and *bd*.

4. Experiment with irregular broken-line patterns, but limited by the parallels *bd* and *ef*, and compare them with regular patterns.

### Intersecting Lines.

**116.** When two lines *end* at one point, so that neither of them crosses the other, as CA and BA meet at A in Fig. 36, they form *one* angle, CAB, in the sense in which an angle is commonly understood; that is, as being *less than* 180°, or than two right angles. The other angle of *more* than 180°, indicated by the arc around the letter A, is very seldom meant.

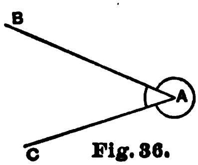

Fig. 36.

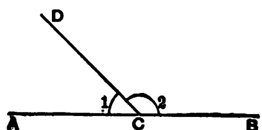

Fig. 37.

**117.** When, as in Fig. 37, one line, DC, *meets* a second line, AB, which *crosses* CD, *two* angles, each less than 180°, are formed. *Such a pair of angles* are called *adjacent angles*, since they border upon each other along the common boundary, CD.

**118.** *Each angle* of a pair of adjacent angles is called the *supplement* of the other, or an angle added to make a pair of adjacent angles.

Having given the measure of *one* of two adjacent angles, the other one is found by subtracting the given angle from 180°.

In other words, *the supplement of any angle is 180° minus that angle.*

**119.** We have seen that there may be many different acute angles and many different obtuse angles, but that all right angles are alike or of one size. Whatever is added to an acute angle to make a right angle is accordingly called the *complement* of that angle, or what makes it complete as a right angle, the standard of comparison for other angles (23).

Thus, in Fig. 38, if CAB is a right angle, CAD is the *complement* of BAD, and BAD is the *complement* of CAD.

Since these terms are often used, let it be remembered that *an angle and its complement* make 90°, or *one* right angle; and *an angle and its supplement* make 180°, or *two* right angles.

**120.** When each of two lines crosses the other, as in Fig. 39, *four angles*, $a$, $b$, $a'$ (read $a$ prime), and $b'$, are formed, all having the same vertex.

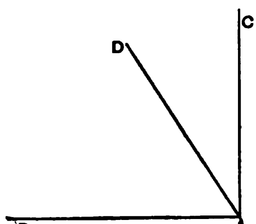


Fig. 38.

The angles $a$ and $a'$, on opposite sides of the common vertex, are called *vertical angles;* $b$ and $b'$ are also vertical angles.

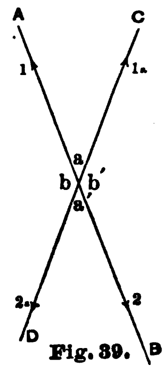


Fig. 39.

**121. Theorem.** — *Vertical angles are equal.* This follows directly from the definition of an angle; viz., the difference in direction between two lines. For suppose two persons at 1 and 1$a$ to move in the direction of the arrows 2 and 2$a$. As they approach the vertex, their paths form with each other the angle $a$; and as they both pass beyond the vertex, each keeping his own line, each continues the *same direction* as before, and on the *same line.* Hence the difference of direction between their paths remains the same, or the angle $a'$ is equal to the angle $a$.

In like manner, it can be shown that $b = b'$.

**122.** The theorem just given is an example of a short and simple proof, directly from first principles, and so may be compared to a straight line as the shortest and simplest path from one point to another. But the truths of geometry are so linked together, that different roads to the same result are often found by comparing those truths. As a simple illustration of this fact, see another demonstration of the last theorem, as follows: —

The angles $a$ and $b$ together are equal to two right angles, because they fill all the space on one side of a straight line, AB. Likewise $a'$ and $b$ are equal to two right angles; that is,

$$a + b = 2 \text{ R. A.},$$

and

$$a' + b = 2 \text{ R. A.}$$

Then, by axiom 3d,

$$a + b = a' + b.$$

But, by subtracting $b$ from both sides of this equation, we have, by axiom 5th,

$$a + b - b = a' + b - b,$$

or

$$a = a'.$$

**123.** By comparing the two proofs that vertical angles are equal, we see that in the *first* we have been reasoning on the *ideas* of *form* (*straight* lines) and *position* (their direction), and in the *second* one, on the idea of *size* (the different angles compared as to their size).

This distinction runs through the whole of geometry; so that, in fact, large books on the "geometry of position" have been written.

We shall from time to time point out theorems of both kinds. Both are equally useful, since we have as much to do in practical affairs with the *forms* of things as with the *sizes* or measure of things.

**Practice.**—1. When we speak of *the* angle which two lines make with each other, where does each line end?

2. When the line AB, Fig. 36, turns to the right about the fixed point A as a centre, what is the effect upon the size of the angle BAC?

3. When AB comes, by turning about A, to have the same direction as AC, what angle do AB and AC make with each other?

4. When two lines form only two angles, each *less* than 180°, how do they meet?

5. What are such a pair of angles called?

6. What is their sum in degrees and in right angles?

7. When is one angle the *supplement* of another? When is it the complement?

8. Having given an angle *in degrees*, how is its supplement and complement found?

9. Having given an angle *on paper*, construct its supplement and its complement?

10. What are the supplements and complements of angles of 5°, 18°, 24°, 30°, 45°, 60°, 72°, 90°, 120°? Construct any three of each of these on paper.

11. When two lines cross each other, how many angles are formed?

12. Of how many different sizes are these angles?

13. How many pairs of equal angles? how situated? and what are they called?

**14. If one of a pair of vertical angles is acute or obtuse, what must the other be?**

**15. If one of two adjacent angles, supplements of each other, is a right angle, what must the other be?**

**16. If one of the four angles made by two intersecting lines is 90°, what must the rest be?**

**124. Theorem.**—When two lines are perpendicular to each other, every point of each is at equal distances from those points of the other which are equidistant from their point of intersection.

Let AB and CD be perpendicular to each other at O. Then, if *a* and *b* are at equal distances from O, any point on AB, as A, or *c*, or B, will be at equal distances from *a* and *b*. This follows at once from the nature of the independent directions of AB and CD, which make them perpendicular to each other (23). For, in moving from O along AB, we make no progress in the direction CD or DC. Hence at A, or *c*, or B, we can be no nearer to *a* than to *b*, or to *b* than to *a*. Hence *ac* and *bc* are equal.

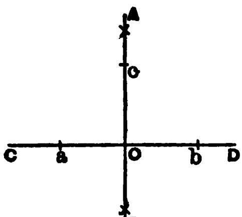

Fig. 40.

Also, if *ac* and *bc* are equal, and AB perpendicular to CD, *a*O = O*b*.

This almost self-evident theorem enables us to solve very simply the two following useful problems with compasses, as well as with a ruler and right-angled triangle, as explained in (74).

**125.** *First Problem.*—Let it be required to draw a perpendicular to a given line, CD, at any point, O, *on* the line (Fig. 41).

1°. Set off on CD, from O, any two equal distances, O*a* = O*b*.

2°. With *a* and *b* as centres, and any one radius greater than O*a*, describe arcs, and note their intersection, *c*.

3°. Draw *c*O, which will be the required perpendicular.

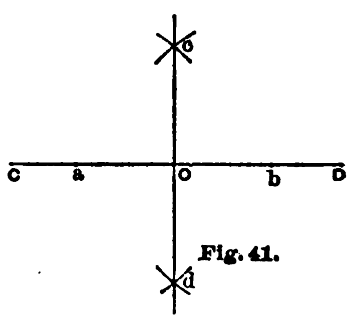

Fig. 41.

Similar arcs drawn at *d* will be a check on any error made in drawing the former arcs, since *c*, O, and *d* should all be in the same straight line.

**126.** *Second Problem.* — Let it be required to draw a perpendicular to a given line, CD, from a given point, A, *not* on the line (Fig. 42).

1°. With the point A as a centre, and any sufficient radius, draw an arc, *bc*, cutting CD at *b* and *c*.

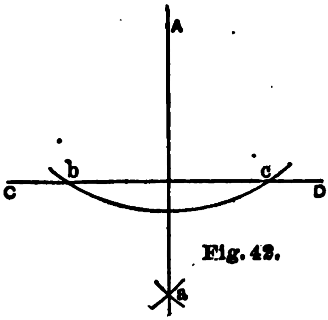


Fig. 42.

2°. With *b* and *c* as centres, proceed, as in the last problem, to draw two arcs having the same radius, and meeting as at *a*.

3°. Draw A*a*, which will be the perpendicular required.

**Practice.** — 1. In Fig. 40, how do *ca*, or *cb*, or B*a* compare in length with *c*O or BO?

2. What is the shortest distance from a point to a line?

3. Why would arcs of less radius than O*a*, and centres *a* and *b*, be of no use in Fig. 41?

4. In Fig. 42, why would the arc with centre at A be useless if it did not cut CD?

5. Draw Fig. 42 with A below CD.

6. In Fig. 40, construct the perpendicular to AB from C or D.

7. In Fig. 40, draw the perpendicular to AB at O

8. In Figs. 41 and 42, place the *given* line in any inclined position.

9. In Fig. 40, if AO is greater than BO, which will be the longer, A*a*, or B*a*?

10. Solve the two problems of perpendiculars on the ground, with radii ten feet or more in length, using a stout cord to draw the arcs.

Exactly the same construction that is shown in Fig. 42 serves to bisect the angle whose vertex is A, and whose sides are A*b* and A*c*. It is, therefore, here applied to that purpose.

**127. Problem.** — *To bisect a given angle.*

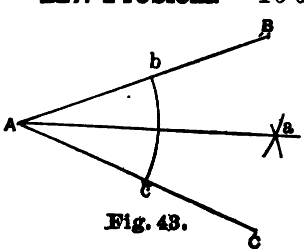


Fig. 43.

Let BAC (Fig. 43) be the given angle.

1°. With A as a centre, and any convenient radius, describe an arc, *bc*, to give points *b* and *c* at equal distances from A. Then,

2°. With *b* and *c* in succession as centres, and some *one* convenient radius, describe arcs intersecting each other, as at *a*.

3°. **Draw the line A*a*, which will bisect the angle BAC.**

For, as A*b* = A*c*, and as every point of A*a* is, by (124), equally distant from *b* and *c*, the point *c* would fall on *b* if *a*AC were folded over about A*a* upon *a*AB; that is, AC would fall on AB, which could only happen when the angles *a*AC and *a*AB are equal.

**Practice.**—1. Bisect any acute angle.
2. Bisect a right angle.
3. Bisect any obtuse angle.

**128.** Curious as it may seem, when it is so easy to *bisect* an angle, there is no known simple way to *trisect* one—that is, divide it into three equal parts—by constructions with straight lines and circular lines alone. A right angle, however, can be trisected.

**129.** In proceeding to consider the intersection of a straight line with more than one other straight line, we will first consider that of a straight line, as EF, Fig. 44, with *two parallel lines*, as AB and CD.

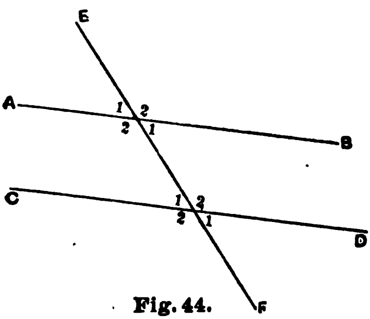

Fig. 44.

This case is interesting as a simple example of how much that is important for use in other cases can be learned by merely looking at the figure (that is, by *inspection*, as it is called) in the light of what is already known.

*First.*—If one line crossing another makes *four* angles, one line crossing two others at different points must form *eight* angles.

*Second.*—Since AB and CD, being parallel, lie in the *same direction*, the difference in direction between AB and EF is the same as the difference in direction between CD and EF. But these equal differences of direction are the angles which AB and CD make with EF. These angles are therefore equal. We may accordingly, if we please, regard CD, and the four angles made by it with EF, as merely a new position, in the same direction, of AB, and its four angles with EF.

*Third.*—Then, since there are but two different sizes of angles made by AB with EF, the same two sizes of angles will be made by CD with EF; that is, the *eight angles* formed by the intersection of a line with two parallels are in *two sets* of four equal angles each, as is indicated by the figures 1 and 2. All the angles marked 1 are obviously equal, and so are those marked 2.

*Fourth.* — The sum of every angle marked 1 with every angle marked 2 is 180°.

**130.** On account of their frequent occurrence in studying other figures, the various pairs of angles in Fig. 44 have, for convenience, received certain names.

Indicating them now by the letters placed in the angles, Fig. 45, *a*, *b*, *g*, *h*, are *external* (that is, outside) angles; *c*, *d*, *e*, *f*, are *internal* (that is, inside) angles; *a* and *e*, and others whose positions are alike, are *corresponding* angles; *d* and *e*, and other like pairs, are *alternate* angles; *b* and *h*, and other like pairs, are *opposite external* angles; *d* and *f* are *opposite internal* angles; *b* and *e*, and other like pairs, are sometimes called *conjugate* angles, or *external-internal* angles.

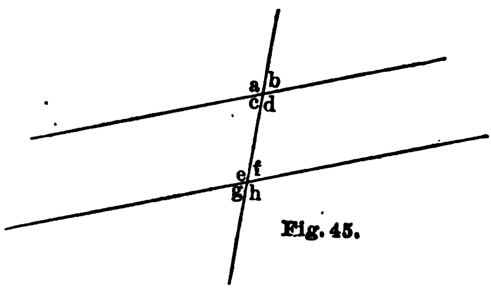

Fig. 45.

**131.** Returning now to the particulars begun in 129, —

*Fifth.* — It is very evident from Fig. 44, where the equal angles are similarly numbered, that, if any one of the eight angles is a right angle, all the rest must be such; that is, —

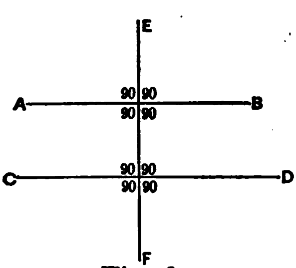

Fig. 46.

*Sixth.* — If a line, EF, Fig. 46, is perpendicular to either one of two parallels, it must be perpendicular to both of them; and

*Seventh.* — If two lines are both perpendicular to a third line, they must be parallel to each other (see Fig. 46); and, in general, —

*Eighth.* — When two lines make equal angles with a third line, they are parallel, for the equal angles show that the difference of direction between the two lines and the third line is the same for both lines: and this shows that the two lines have the same direction; that is, that they are parallel.

**Practice.** — 1. When a line crosses two others at different points, how many angles are formed?

2. If the two lines are parallel, how many different sizes of angles are formed?

3. How many pairs of equal angles are formed?

4. What is meant by learning by inspection?

5. What is an angle?

6. How much difference of direction have parallel lines?

7. How, then, must the angles compare which they make with a third line?

8. When a line meets two parallels, what is the sum of every pair of unequal angles?

9. Draw Fig. 44 with the parallels in the direction of the third line, and the third line in any other direction. Then number the four smaller equal angles 1, 2, 3, 4, and the larger ones 5, 6, 7, 8.

10. Fill out the following blanks with the proper numbers, from the last figure: —

The *corresponding* angles are
The *alternate* angles are
The *pairs of vertical* angles are
The *conjugate* angles are
The *external* angles are
The *internal* angles are
The *opposite external* angles are
The *opposite internal* angles are

11. If $a$, Fig. 45, is 105°, what will each of the other seven angles be?

12. If $a$ is 90°, what will the other seven angles be?

13. How many of the angles in any figure like Fig. 45 need be measured with a protractor in order to know the size of all of them?

14. If in the eight angles there were more than *two* different sizes, what would that show as to the two lines AB and CD.

15. For every additional line parallel to AB, and crossing EF, how many *more angles* would be formed?

16. How many *more different-sized angles* would be formed?

17. In Fig. 44, draw a parallel to EF. How many angles in all will then be formed?

18. How many different sizes of angles will be formed?

19. What is the *least* number of acute angles that can occupy 360°, or all the space about a common vertex?

20. What is the *greatest* number of *obtuse* angles that can fill all the space around a point as their common vertex?

# CHAPTER IV.

## TRIANGLES.

### Kinds and Simple Properties.

**132.** As soon as we begin to study *three* lines which all meet each other, we see that they must either all meet at the *same point*, or else at *three different points*.

In the former case, shown in Fig. 47, they form three different angles, ABC, ABD, CBD, at *one* point, B.

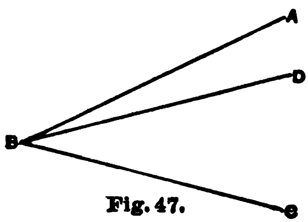


Fig. 47.

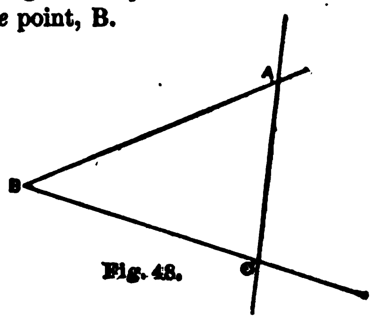


Fig. 48.

In the latter case, shown in Fig. 48, they form three different angles, A, B, and C, at three different points. Also, in this case, the lines completely enclose a space, ABC, called a *tri-angle*,— commonly written thus, *triangle*, — and meaning a figure of three *angles*, as well as three *sides*.

In the former case the lines do not enclose a space.

It is also evident, from Fig. 48, that no less than three lines can enclose a space.

**133.** Next, it is evident, from Fig. 49, that just because a triangle has the fewest sides that a figure can have, any figure composed of more than three sides, as ABCDE, can always be divided into triangles.

**The simplest parts into which any thing can be divided, or of which it is composed, are called its *elements*.**

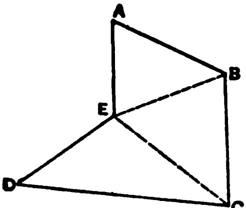

Fig. 49.

**Because the elements of any thing are its simplest parts, it is natural to study objects by studying their elements.**

**The triangle, being the simplest possible plane figure, is the *elementary figure* of which other figures are composed. The *study of plane figures*, therefore, naturally begins with the study of the *triangle*.**

**Practice.—1. In how many and what ways can three lines (of unlimited length in both directions) meet?**

**2. Draw three limited lines so as to form *two* points of meeting.**

**3. Draw three lines meeting as in Fig. 48, but in different positions, forming different-shaped angles.**

**4. Divide Fig. 49 into triangles by lines drawn from B or from C.**

**5. How many sides and how many angles has every triangle?**

**134.** By drawing three intersecting lines, as in Fig. 48, in many different positions, we shall find that there are many differently shaped triangles. The same can be learned by dividing up many differently shaped figures, as in Fig. 49, into triangles.

**135.** To distinguish the angles and sides of triangles conveniently, we say that each *angle*, as B, Fig. 48, is *included* between the sides, as AB and CB, which form it, and is *opposite* to the remaining side, as AC. Any *side*, as AB, is also said to be *opposite to the angle*, as C, which is included between the two remaining sides, AC and BC.

**136.** Moreover, we shall easily enough be satisfied, by simply drawing many differently shaped triangles, that in any triangle the *largest angle and the longest side are opposite to each other.* This, however, is easily proved, as follows:—

**137. Theorem.**—*In every triangle the largest angle and the longest side are opposite each other.* Since a straight line is the shortest distance between two points just because it proceeds in but *one* direction between those points, it follows at once that the *less* it is departed from by other paths between the same points, the *less* will the distance between the points be increased by that departure; that is (Fig. 50), the route, *abc*, between *a* and *c*, is necessarily shorter than *adc* is, because *abc* departs *less* from the shortest route, *ab*, than *adc* does.

It can now be seen that the longest side and largest angle of a triangle must be opposite to each other; for the nearer *d* is carried to *ac*, as at *b*, the longer the side *ac* becomes *as compared with ab* and *bc*. But, at the same time, the angle at *b* is *larger* the nearer *b* is to *ac*, because the nearer to a straight line *abc* comes, the nearer to 180° will the size of the angle *b* become; for when *b* reaches *e* on *ac*, *aec* is an angle of 180°.

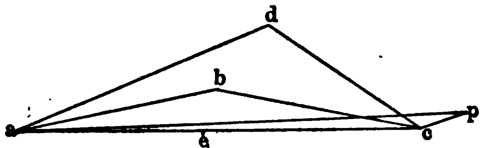

Fig. 50.

This, however, supposes that *b*, as in the figure, is nearer to both *a* and *c* than *a* and *c* are to each other. If *b* were placed at *p*, for example, the above reasoning would apply to *c*, and the longest side, *ap*, would be found opposite to the largest angle, *acp*.

**138.** It will now aid the further study of the different *forms* of triangles to understand the following theorem, in which an angle is regarded, as in (95), as formed by a line sweeping around one of its points as a centre.

**Theorem.** — *The sum of all the angles in any triangle is equal to two right angles.* Let ABC, Fig. 51, be any triangle. We will suppose it to be on the ground, since neither its *size* nor its *position* will affect its *form*. It can be of any possible shape anywhere. Then suppose yourself standing at A, with your back to C, and hence facing in the direction CA, as shown by the arrow W, which we will suppose to point *west*. Then turn to your left till your back is towards B. You will then have turned through the angle WA*a* = CAB, and will be facing in the direction BA, as shown by the arrow *a*. Then turn again so as to face in a direction, A*b*, parallel to BC, which will be the same as if you had stepped backwards from A to B while facing in the direction of the arrow *a*, and had then turned through the angle B so as to face C. Finally turn so as to face C, which will be the same as stepping from B to C, and then turning through the angle

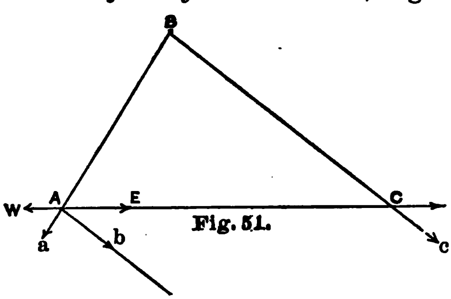

Fig. 51.

C, so that your back will be towards A, as shown by the arrow E, which then points *east*. Thus you see, that by standing at A and facing W, and then turning to face $a$, then $b$, then E, you have turned through three angles equal to the three angles of the triangle; and in doing so, you have turned just half round, or through two right angles. Hence the three angles of a triangle are equal to two right angles, or 180°.

*Second Proof.* — One may turn entirely around, or through four right angles, either at one point, or by walking around a plane figure to his first position. Thus, one standing in one corner of a room, and facing the next corner, may turn entirely around in that corner so as to face the same next corner again. Or one may walk around the four sides of the room, turning one-quarter round at each corner, until he stands in the first corner again, facing the same way as at first. In either case the person has turned once entirely around, or through four right angles. We shall employ this truth in the following proof of the last theorem: —

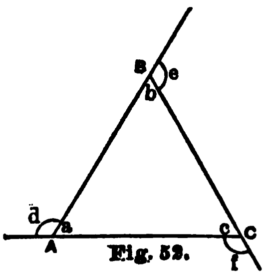


Fig. 52.

Let ABC, Fig. 52, be any triangle, on the ground. Suppose a person to stand at A, facing B. Walking to B, he will there turn through the angle $e$ to face C. Walking to C, he will there turn through the angle $f$ to face A. Walking to A, he will finally there turn through the angle $d$ to face B again. He has then turned once entirely around; that is, the angles

$$d + e + f = 360°, \text{ or four right angles.}$$

But, by (122),

$$\begin{array}{ll} \quad d+a & = 2 \text{ R. A.,} \\ \quad e+b & = 2 \text{ R. A.,} \\ \quad f+c & = 2 \text{ R. A.; hence, taking the sum of these, we have} \\ \hline (d+e+f)+(a+b+c) & = 6 \text{ R. A. Subtracting} \\ (d+e+f) & = 4 \text{ R. A., we have left,} \\ \hline \qquad a+b+c & = 2 \text{ R. A.} \end{array}$$

*Third Proof.* — The two preceding proofs were made by the aid of the idea of *motion*, — a method which is often interesting and useful. We will next give the usual proof, which consists in comparing the sizes of certain fixed parts of a diagram.

In Fig. 53, draw through any vertex, as C, a line, CD, parallel to the opposite side, as AB. Then the angle *a* = angle A (129),

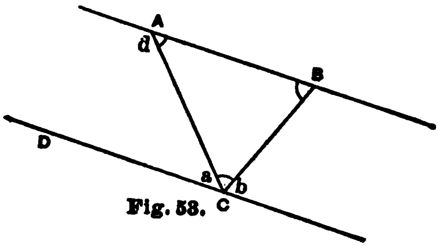


Fig. 53.

as will be more plainly seen by extending the line BA as in the figure. Likewise *b* = B. Then

$$\text{angles } a + C + b = \text{angles } A + C + B.$$

But $a + C + b = 2$ right angles, being all the space around C and on one side of a straight line. Hence, also, $A + B + C = 2$ right angles, which was to be proved.

**139.** The angle *d*, Fig. 53, included between one side, CA, of a triangle, and an adjacent side, BA, produced, is called an *exterior* angle. Thus, *d*, *e*, *f*, Fig. 52, are exterior angles. The last theorem now naturally leads to the following: —

**Theorem.** — *An exterior angle of a triangle is equal to the sum of the two opposite interior angles.*

The angles *d* and CAB, Fig. 53, are together equal to two right angles, being adjacent angles, or all the angular space around their common vertex, and on one side of AB.

But A, B, and C are also together equal to two right angles. Hence, subtracting, in both cases, the angle at A, which is common to both of these sums, we have left the exterior angle *d* = the sum of the two opposite interior angles B and C.

**140.** A truth such as those just learned (138, 139), is learned by something more than merely *looking at the figure*. It is learned by starting a thought about the triangle which leads to another thought, and that to another, until we reach a stopping-place by finding out something about the triangle which we did not know before.

What is learned by merely looking at a figure is said to be learned by *inspection* (129).

What we learn by a succession of thoughts, each growing out of the last, is called a *theorem* (111); and the succession of thoughts make what is called the *demonstration* of the *theorem*, or the *plain showing* that the theorem is true.

**Practice.**—1. In what ways can we learn that there are many different forms of triangles?

2. What is meant by an included angle?

3. What by an opposite angle?

4. What is the relative position of the largest side and largest angle?

5. Prove, by walking around a triangle on the floor or on the ground, and turning at each angle by an amount equal to that angle, that the three angles together are equal to two right angles.

6. What is learning by inspection?

7. What is a theorem?

8. Tell what you can learn about a triangle by simple inspection.

9. Show, by a figure on the blackboard, that the three angles of a triangle are equal to two right angles. (The figure to represent what is actually done in question 5 above.)

**141.** If *all three* angles together of any triangle make just two right angles, *any two* of them must be less than that, so that no triangle can have more than *one right angle*. Still less can any triangle have more than *one obtuse angle;* that is, in every triangle *two of the angles must be acute*.

If *two* angles of a triangle are known, the *third* is 180° minus the sum of the known angles; that is, if $A + B + C = 180°$, $C = 180° - (A + B)$. Hence, if two angles of one triangle are equal to two angles of another triangle, the remaining angle in each must be the same.

**142.** What we have now learned enables us to *name every form of triangle* by means of its *most marked difference from other triangles*, whether that difference be in the *sides* or in the *angles*. Thus,—

*First*, as to the *sides:*—

1°. A triangle *all* of whose *sides* are equal is *equilateral* (69); ACD, Fig. 18.

2°. A triangle *two* of whose *sides* are equal is *isosceles* (69); ABC, Fig. 17.

3°. A triangle *no two* of whose sides are equal is *scalene* (Fig. 52).

*Second*, as to the *angles:* —

1°. A triangle *all* of whose angles are *acute* is an *acute-angled triangle* (Fig. 52).

2°. A triangle *one* of whose angles is a *right angle* is a *right-angled triangle*, also more briefly called a *right triangle* (Fig. 54). AC, opposite B, the *right* angle, is called the *hypothenuse*.

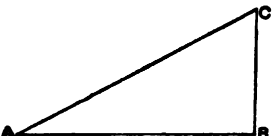


Fig. 54.

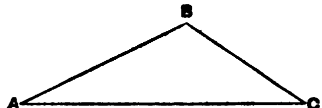


Fig. 55.

3°. A triangle *one* of whose angles is *obtuse* is an *obtuse-angled triangle* (Fig. 55).

**Practice.** — 1. In Fig. 53, make the proof by drawing a parallel to AC through B.

2. In Fig. 54, which *side* is opposite to the angle A? the angle B?
3. In the same figure, which *angle* is opposite BC?
4. Which *sides* include the angle C? the angle B?
5. Could B and C *both* be *obtuse?* Could both be *right angles?*
6. Draw a *right-angled isosceles* triangle.
7. Draw an *obtuse-angled* isosceles triangle.
8. Every *equilateral* triangle is what kind of an *angled* triangle?
9. In Fig. 52, a *perpendicular* from any angle (corner, vertex) to the opposite side divides the triangle into two triangles of what kind?
10. How many such perpendiculars can be drawn *within* either a right-angled or an obtuse-angled triangle?
11. What is the longest side of a right triangle called?
12. *Where* is it?
13. If two of the angles of a triangle are 42° and 77°, what is the remaining angle?
14. If one angle of a triangle is 48°, what is the sum of the other two?

### Symmetry.

**143.** Symmetry in geometry consists in being divided by some straight line into parts, which, when folded together about that line, exactly coincide. Such a line is called an *axis of symmetry*, or a *centre line;* and the figure which it divides, as just described, is said to be symmetrical with respect to that axis.

A butterfly affords one of the most beautiful and familiar natural illustrations of symmetry. Its wings when folded together appear as one; when spread, they appear as equal and opposite halves divided by the centre line of the creature's body. Leaves of plants are also often symmetrical with respect to their central rib or line.

**144.** Many geometrical truths are most easily and beautifully made clear by the aid of the idea of symmetry.

Thus, let ABD, Fig. 56, represent *two* equal right-angled triangles laid one upon the other so that they coincide, and with their right angles at D, and the sides AD placed against the line OX. Now, any one who has ever seen a circular saw, or even a grindstone or a bicycle, in motion, knows well enough, when they are

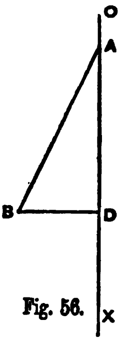


Fig. 56.

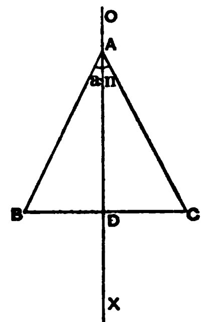


Fig. 56(a)

*seen exactly edgewise*, the saw, or either flat face of the stone, or the rim of the bicycle, appears as a straight line perpendicular to the axis on which it revolves. Hence, when *one* of the triangles at ABD, both now in the paper and at the left of OX, is revolved 180° over to the right about the axis OX, it will fall in the paper as shown at ACD, Fig. 56*a*. Hence AC = AB, for these both coincided at AB in Fig. 56. For the same reason, the angle *a* = angle *n*, both the angles at D are right angles, and CD = BD, and the figure ABC is an *isosceles*, or symmetrical, triangle. It would have been *equilateral* if BD had been equal to half of AB.

**145.** By examining the triangle ABC in the light of the idea of symmetry, we easily learn many simple geometrical truths: —

1°. AD is *first* perpendicular to BC; *second*, it bisects BC; *third*, it bisects the angle A. These three truths stand together.

2°. If CD = BD, then AC = AB (124).

3°. If CD were *less* than BC, C would fall between B and D, as at *c*, Fig. 57, and Ac = AC would therefore be shorter than

AB. This is also evident on comparison of both with the *independent directions* (23) AD and DB; for, in going from A to B, we progress as far in the direction AD as if we had moved on AD, and also as far in the direction DB as if we had moved over DB. But in going from A to *c* we make the *same* amount of progress, AD, in the direction AD, as before, but only the *less* progress, D*c*, in the direction DB. The path A*c* must therefore be shorter than AB.

4°. In like manner, if BD, Fig. 56, were greater than CD, as it is in Fig. 57, AB would be longer than AC, as it is in Fig. 57.

5°. The nearer C is to D, the shorter AC must be; and when it is nothing, or when C falls on D, AC becomes the *perpendicular*, AD, which is the *least distance* from A to BC. This is also evident from the idea of progress in the direction of the independent directions AD and DB; for, when we move over AD only, we make *no* progress in the direction DB, and hence must travel a shorter distance than if we *also* made any progress, however small, in the direction DB (112).

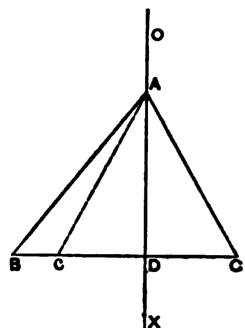

Fig. 57.

That is, *of all lines from a point to a given line, the perpendicular is the shortest;* lines equally inclined to the given line are equal; and the *less* the angle they make with it, the *longer* they are.

6°. As none of the foregoing principles depend on the position on OX of the point A, they are true wherever A is taken on OX.

### Base and Altitude, etc.

**146.** The *base* and *altitude* of a triangle are a connected *pair of lines* of frequent use in the study of triangles, as will be seen, and they should therefore be well understood.

A triangle having *three sides* and *three angles, each side in succession* can, as found convenient for any purpose, be called the *base;* and *then* the *altitude*, or height, of the triangle will be the *perpendicular to that side* (or to that side prolonged) from the *opposite angle*.

**147.** According to the descriptions just given, and both in the given acute-angled triangle, ABC, Fig. 58, and in the given obtuse-angled triangle, Fig. 59, —

1°. When AB is the *base*, Cc is the *altitude*.

2°. When AC is the *base*, B*b* is the *altitude*.

3°. When BC is the *base*, A*a* is the *altitude*.

Note, also, that the intersection, *d*, of the altitudes is *inside* of the *acute*-angled triangle, because *all the altitudes are wholly within* the triangle. But the like point, *d*, in Fig. 59 is *outside* of the *obtuse*-angled triangle, because *two* of the altitudes fall outside of the triangle, which are those from the two acute angles.

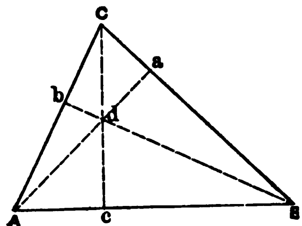


Fig. 58.

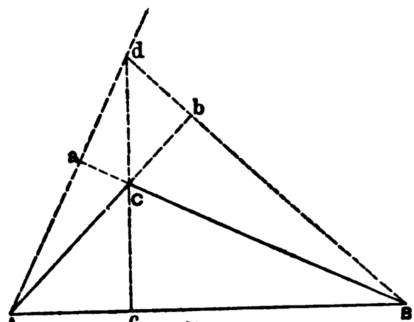


Fig. 59.

**148.** It is a curious group of facts, the proofs of which can be found in larger works, that not only (1) do the *three altitudes* of every triangle intersect in one point, but the same is also true (2) of the bisecting lines of the three angles, and (3) of the lines from each angle to the middle of the opposite side, and (4) of the perpendiculars to the three sides at their middle points. We mention these facts here as affording some excellent examples for practice in perfectly exact construction. The centres of the arcs drawn in making the construction must be exactly on the corners of the triangles. All lines must be fine, and intersect sharply, and be drawn exactly through the points through which they are intended to pass.

The smaller the triangles, the more difficult it will be to make the three lines in each of these cases intersect at exactly the same point, since any mechanical errors — that is, errors in the use of the instruments — will then be larger in proportion to the figure than if the figure were larger.

**Practice.** — 1. Repeat Fig. 58 in a different *position*, and with different letters.

2. Repeat Fig. 59 with AC horizontal.

3. Repeat Fig. 59 with BC horizontal.

4. Let the given triangle be equilateral.

5. Let the given triangle be isosceles.

6. When *d* is *inside* of the given triangle, what kind of a triangle is it?

7. In what kind of triangle does *d* fall *outside*?

8. What is meant by the altitude of a triangle?

9. In the triangular end of a roof, what lines would you call the base and altitude?

10. Make *two* groups of figures of *four* each, one for each of the four cases described in 148, making all the triangles of one group *acute*-angled, and all those of the other group obtuse-angled.

### Comparison of Triangles.

**149.** Triangles may be compared either with respect to their *size* or their *shape*. There will therefore be *four cases*: —

1°. They may differ in both shape and size, as in Fig. 60.

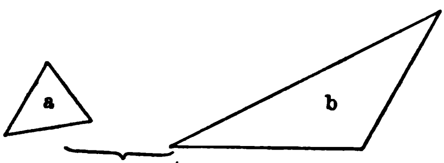


Fig. 60.

2°. They may agree in *both*, as in Fig. 61. Such triangles are called *equal*.


Fig. 61.

3°. They may agree in *shape only*, as in Fig. 62. Such tri-

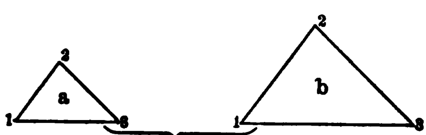


Fig. 62.

angles are *similar*, and they are so because the angles 1, 2, 3, of the one are equal to those, 1, 2, 3, of the other.

4°. They may agree in *size only*, as ABC and ABD in Fig. 63. Such triangles are called *equivalent*, or of equal value, though not wholly alike.

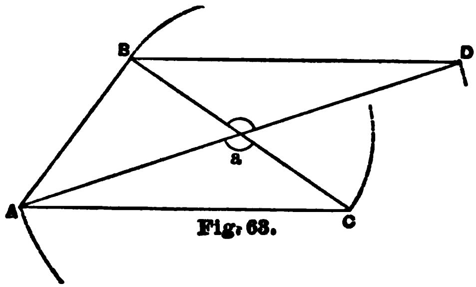


Fig. 63.

In this figure, the triangle B*a*D is made equal to A*a*C. Then, by adding the same triangle, A*a*B, to each, we form the equivalent though differently shaped triangles ABD and ABC.

### Excess or Difference, and Ratio.

**150.** If A represents a line two feet long, and B one three feet long, and we add two feet to each, the first will be four feet, and the second five feet long, and the *excess* of each new line over its former length will be two feet. But note that the line A was at first *two-thirds* ($\frac{2}{3}$) of the line B, while the line 4 is *four-fifths* of the line 5; and $\frac{2}{3}$ and $\frac{4}{5}$ are the *ratios* of the two lines in the two cases. From this example we see, that, by making *equal differences* in each of two things, we alter their *ratio*. *Ratio* is the *number of times* that one thing is larger than another, and in common speech is often called the proportion of one thing to another.


Fig. 64.

**151.** Again, if line A be doubled, it will be four feet long; and if line B be doubled, it will be six feet long. Line A is then made two feet longer than before, while line B is made three feet longer than before. But note, that the line A, after being doubled, is four-sixths ($\frac{4}{6}$) of the line B when doubled. But $\frac{4}{6} = \frac{2}{3}$, which is the same *ratio* that the lines had at first. From this we see that when the lengths of lines are altered so as to preserve their *ratio* unchanged, unequal differences are made in them; that is, their lengths are altered unequally.

**152.** These simple principles enable us to understand more exactly what is meant by *similar* triangles. The sides of the triangle *abc*, Fig. 65, are *bc* = 2, *ac* = 3, *ab* = 4, measured in sixths of an inch. The sides of the triangle ABC, Fig. 66, are each double that of the corresponding side of *abc*, or BC = 4, AC = 6, and AB = 8; and we *see* that the triangles *abc* and ABC are of the same shape, or similar. We may consider ABC as

**Fig. 65.**

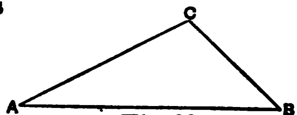

**Fig. 66.**

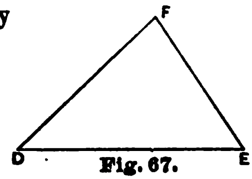

**Fig. 67.**

*abc* magnified, as if seen through a magnifying glass. But it is evident enough, that unless the glass magnified all parts uniformly, — that is, made each part the *same number of times* larger than it was before, — the image of the object would be out of shape. Thus, if we *added* equally to the three sides of *abc*, as by making

$$bc = 2 + 3 = 5,$$

$$ac = 3 + 3 = 6,$$

and

$$ab = 4 + 3 = 7,$$

we should find the triangle DEF, Fig. 67, where EF = 5, DF = 6, DE = 7; and this triangle is evidently of very different shape from *abc* and ABC, which are of the *same shape*, or *similar*, though of different size.

**153.** But though we *see* that what has now been shown is true, the question remains, *Why* must it be so? This question can be easily answered.

Similar figures are of the *same shape*, but unequal in size. Let us now examine and see what makes them of the same shape. It is having their angles equal, taking them in the same order. A = *a*, B = *b*, C = *c* (Figs. 65 and 66). This being so, it next follows, that, as *angles fix the direction of lines*, if AB be made parallel to *ab*, and angle A at its left, as *a* is at the left end of *ab*, AC must be parallel to *ac*, and BC parallel to *bc*.

Now let us see what follows next.

If a line, CD, Fig. 68, be divided into equal parts, and parallels, *ab*, *dh*, etc., be drawn through the points of division, these parallels will divide equally any straight lines which cross them all. For, first, as the parallels, *ab*, *dh*, etc., are equidistant, any two of them, as *gk* and AB, may be thought of as merely a new position of some other two, as *ab* and *dh*. Second, the distance between two parallel lines evidently depends on the *direction* in which they are crossed. Putting these two principles together, it is sufficiently evident, that if parallels which are at equal distances apart are crossed in one and the same direction, the distances between these parallels and in that direction will be equal, as $ad = dg = gA$.

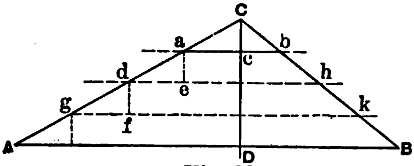

Fig. 68.

**154.** Thus, we learn that if two triangles are similar, and if the side P of one of them is 3, or *any* number of times the side *p* of the other, then the sides Q and R of the one will be the *same number* of times the corresponding sides *q* and *r* of the other. This is what is meant when it is said that in similar triangles the corresponding sides are proportional.

Since all other figures can be divided into triangles, what is true of similar triangles is true of all similar figures.

Note, also, that by corresponding sides is meant exactly those whose adjacent angles in one figure are equal to those in the other, as *ab* and AB are corresponding sides, because the angles *a* and A are equal, and the angles *b* and B are equal (Figs. 65, 66).

**155.** One of the most interesting and useful applications of the principle of similarity is the following: —

**Theorem.** — *The triangles formed by the perpendicular from the right angle to the hypothenuse of a right-angled triangle are similar to each other and to the whole triangle.*

ABC, Fig. 69, is a right-angled triangle. BD is the perpendicular from the right angle, B, to the hypothenuse (142), AC. Then, ABD, BCD, and ABC are all of the same shape, or *similar*.

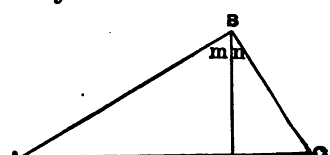

Fig. 69.

*First.* — In ABC and ABD, both have the same angle, A, and both are right angled, the one at B, the other at D. Hence, by (141), the remaining angles, *m* in ABD and C in ABC, must be equal. Hence, by (153), the triangles ABC and ABD are similar.

*Second.* — In ABC and BCD, both have the same angle, C, and both are right angled, the one at B, and the other at D. Hence, as before, the third angles, *n* in BCD and A in ABC, are equal, and the triangles, having the angles of the one equal to the corresponding ones in the other, are similar.

*Third.* — In ABD and BCD, both are right angled at D, *m* in ABD = C in BCD, and *n* in BCD = A in ABD. Hence these triangles are similar.

**Practice.** — 1. Make two unequal triangles of different shapes.

2. Make two *equal* equilateral triangles.

3. See if you can make two dissimilar equilateral triangles.

4. Make two equal isosceles triangles.

5. Make two similar isosceles triangles.

6. Make two equal right triangles.

7. Make two similar right triangles.

8. On opposite sides of a given straight line, taken as the common hypothenuse for both, make two equal right-angled triangles; also two unequal ones. (Make use of (74).)

9. Having a given acute-angled triangle, make a similar one with sides twice as long.

10. Having an obtuse-angled triangle, make a similar one with sides half as long.

11. Having an isosceles right triangle, into what kind of triangles will it be divided by a perpendicular from its right angle to its hypothenuse?

12. In a triangle of any shape whatever, draw a line parallel to any one of the sides, and limited by the other two sides. How will the partial triangle thus cut off compare in shape with the whole triangle?

**156.** Since all other straight-sided figures are, as already mentioned, composed of triangles, we not only learn their geometrical properties by studying the triangles which compose them, but we learn to construct them by constructing triangles. By constructing a triangle is here meant, *not* drawing any three lines at pleasure so as to intersect and form a triangle, as in Fig. 48, but *completing a triangle* when enough, and only enough, of it is given to enable us to make one, and only one, triangle which will possess the given parts.

**157.** It therefore becomes interesting and important to know *how many* and *what* parts must be known in order that only one, or at most two, triangles can be constructed which shall contain those parts. In other words meaning the same thing, we wish to learn by what given conditions a triangle is *determined;* that is, enabled to be completed so as to have but *one size and form* agreeing with the given conditions.

### Construction of Triangles.

If a triangle is oftenest thought of as a three-sided figure, the condition naturally first thought of is, that if *all three sides* of a triangle are given, the triangle is completely determined. Let us see.

**158. Problem.** — *To construct a triangle, having given the length of its three sides.*

Let AB = 1.4 inches, Fig. 70, be one given side of a triangle, and let the other side beginning at A be 0.8 of an inch, and the remaining side starting from B be 1.2 inches. Then take a distance of 0.8 of an inch in the dividers (fitted with a pencil-point), and A as a centre, and describe a small arc about where we judge that the third angle, C, of the triangle will fall. Then with the third side, 1.2 inches in the dividers, and B as a centre, describe a second arc intersecting the former, as at C. Then draw AC and BC, which will complete the triangle, which will really be the only one which can be made with the three given sides. For, if both arcs be produced downward, they will meet as far below AB as they now do above it, and directly opposite to C. Thus, we shall then merely have the triangle ABC turned upside down. Also, if the *length* AC be taken at B for a *centre*, and BC at A for a centre, arcs will be formed above and below AB (as at D above), and the triangles formed by joining this new pair of points with A and B will form a pair of triangles which will be only the former pair turned right for left. Thus, we can form *four different positions* of what is really but one triangle when its three sides are given.

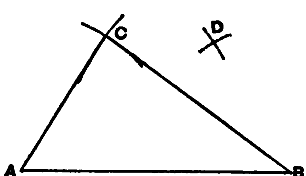

Fig. 70.

**159.** If we make as many triangles as we please with these same three given sides, they will all be like ABC, or will really be the same as the triangle ABC in as many new positions. This is only the same as saying, that, *if the three sides of one triangle are equal to the three sides of another, the two triangles are equal.*

**160.** By introducing the *idea of motion*, we can show that the isosceles triangle is derived from the equilateral triangle, as follows: —

Suppose the equilateral triangle ABC, Fig. 71, in the paper, to turn about the axis $ax$, also in the paper, and perpendicular to BC. Then, if B, moving in a circle perpendicular to the paper, and whose centre is O, turns forward, C will turn backward an equal amount, and OB = OC (145, 1°) will appear shorter, as at $Ob = Oc$, and the equilateral triangle ABC will appear as the smaller triangle $Abc$. The apparent size, $Abc$, of ABC will depend on how far ABC has turned.

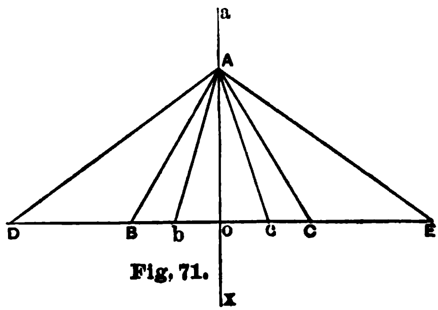


Fig. 71.

Again, suppose ABC, just as it is, to be inclined to the paper; that is, suppose that it is itself the oblique view of some other triangle. Then, when it is turned back about $ax$ until it lies *in* the surface of the paper, it will show its true size, as at ADE.

It is clear that in each of the triangles, $Abc$ and ADE, *two* sides are equal, $Ab = Ac$ and AD = AE (145). They are, therefore, *isosceles triangles;* one of them, $Abc$, *acute* angled, the other *obtuse* angled (142). Isosceles triangles are of great use in geometrical study, as well as of very frequent practical occurrence, as in the gables of roofs and the ornamental caps or tops sometimes seen on doors, windows, and cases.

The unequal side of an isosceles triangle is commonly called its *base*, and the opposite angle is called the *vertex* of the triangle.

**Practice.** — 1. Make a triangle whose sides are 1 inch, $1\frac{1}{2}$ inches, 2 inches.

2. Make a triangle two of whose sides are each $1\frac{3}{4}$ inches, and the third side 0.9 inch.

3. Make a triangle the sum of whose two equal sides is 3 inches, and whose third side is $2\frac{1}{2}$ inches. What kind of triangles are this and the last? and how do they differ, named according to their angles?

4. Construct an equilateral triangle on a line $1\frac{1}{2}$ inches long.

5. Construct an isosceles triangle whose base is 2 inches, and one of whose other sides is $1\frac{1}{4}$ inches.

6. Construct triangles with sides of $\frac{3}{4}$, $1\frac{1}{4}$, and $2\frac{1}{2}$ inches; also of 1 inch, 3 inches, and 1.6 inches. When is the problem impossible?

7. Construct an equilateral triangle each of whose sides is 1 inch.

8. Construct an isosceles triangle each of whose equal sides is $1\frac{1}{2}$ inch, and whose base is $\frac{1}{2}$ inch.

9. Construct, on a base of $1\frac{1}{4}$ inches, five isosceles triangles.

10. Find, by experiment, at what length for the other sides no triangle can be constructed.

11. On a base of 1 inch, make an isosceles triangle whose vertex shall be an obtuse angle.

12. On the same base, make an isosceles triangle whose vertex shall be acute.

13. Between the last two cases, what other angle is possible for the vertex? How many such isosceles triangles can there be on a given base?

14. How many different obtuse-angled and how many acute-angled isosceles triangles can have the same base?

15. Construct a scalene triangle the sum of whose three sides is 4.5 inches, and one of whose sides is 2.1 inches. How does its surface compare in size with that of the equilateral triangle (example 4), in which the sum of the three sides (perimeter) is the same?

16. Draw three separate unequal lines, and then construct the triangle whose sides shall be equal to those lines.

17. Find, by constructing it, what kind of a triangle has for its side 3, 4, and 5 quarter-inches.

**161.** If we oftenest think of a triangle as a three-*sided* figure, we next oftenest think of it as a three-*cornered* figure. Let us then see if a triangle is determined (157) by its three *angles* as well as by its three *sides*. By referring back to 149, we see, that, while having all the *sides* given fixes the *size* of a triangle, having all the *angles* alone given fixes only its *form*. We therefore conclude, that, as is shown more perfectly by Fig. 72, any number of triangles of unequal *size*, though all of the same *shape*, can be made when only the three angles are given.

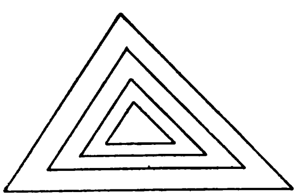

Fig. 72.

We also see, that, as given *angles alone* determine *form*, while given *sides* determine *size* also, we must always have given *such* and *so many* parts of a triangle, one or more necessarily being *sides*, as *will fix both its form and size*, in order to determine the triangle.

**162.** We have seen (159) that triangles are *wholly alike or equal* when the *sides* of any one of them are equal to the sides of the others; that is, the *angles* are then the same in any one as in all the others, making, in all, *six parts*, in respect to which all the triangles are equal. Hence, as the greater includes the less, any less number of parts in each triangle will be equal to the like parts in the other triangles; and we have only to select such as will determine the triangle.

1°. *Three sides*, we have seen, will determine a triangle.

2°. *Two sides and the included angle*, or angle between them, also determine a triangle, because (see Fig. 73) we then know both the *length* and the *direction* (with respect to each other) of these sides, as *ab* and *ac*; and the *third side* can then have no other possible position but to join the fixed extremities, *b* and *c*, of the given sides.

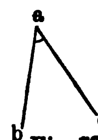

Fig. 73.

3°. *Two sides and the angle opposite* to one of them also determine a triangle; for if, in Fig. 73, we knew the *length* of *ab* and *ac* and the angle at *c*, for example, instead of at *a*, the one only position which *ab* could have would be from *a* to some point on a line making the given angle at *c* with *ac*.

4°. *One side* and its *two adjacent angles*, or

5°. *One side* and an *adjacent angle* and the *opposite angle*, both determine a triangle, because, *first*, the extremities of the *given side* are where the two other sides *must begin;* and, *second*, the *given angles fix the direction of both those sides.* Hence all sides are thus fixed, both in length and direction.

**163.** We now see,—

*First*, That *no other elementary conditions can be had*, because we have used up all which include a side among the given parts;

*Second*, That *in each condition* no fewer parts would answer, because *all of them have been used;*

*Third*, That no more given parts could be used, since those taken in each case serve to fix the length and direction of all the sides; and

*Fourth*, That all the other sets of determining conditions flow from that of three given sides as the simplest and the one naturally thought of first.

If this exercise in thinking (6) is not quite clearly understood, it will be by means of the following construction of each case separately:—

**164. Problem.** — *To construct a triangle, having given two of its sides and the included angle.*

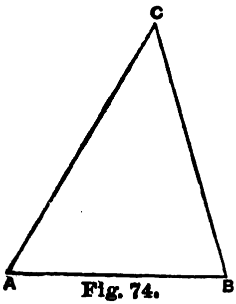


Fig. 74.

Let the given sides be $1\frac{1}{8}$ and $1\frac{1}{2}$ inches in length, and the included angle one of 60°.

Then draw any straight line, and on it lay down AB = $1\frac{1}{8}$ inches. At A make the given angle of 60° by the protractor (104) or the 60° angle of a drawing triangle (69), or as in (100), and on its other side, AC, lay off AC = $1\frac{1}{2}$ inches, the length of the other given side. Then join B and C by a straight line, which will complete the triangle. It is evident that no other *size* or *form* of triangle can be made with these three given parts.

**165.** By introducing the idea of symmetry at this point (143), we can see the *three other positions* which this same triangle might have had, AB still remaining fixed. The angle A might have been made *below* AB, which would have been the same in effect as revolving ABC 180° about AB as an axis; that is, simply inverting it. Again, the *given angle* might have been made at B, and either above or below AB, which would have given the same result as if the former pair of *positions* had been revolved 180° around a perpendicular to AB at its middle point.

**Practice.** — 1. Construct a triangle two of whose sides are $1\frac{1}{4}$ and $1\frac{5}{8}$ inches in length, and whose included angle is 37°.

2. Draw two separate lines of convenient length, and an angle, on the paper, and then make the triangle having these lines and this angle for two sides and the included angle.

3. Construct an isosceles triangle of which the equal sides are each $\frac{3}{4}$ inch, and their included angle 25°; also when this angle is 90° and when it is 160°.

4. Construct a right-angled triangle with sides including this angle of $1\frac{1}{2}$ inches and 1 inch.

5. Draw the three other *positions* of the last triangle afforded by the sides of its right angle as axes of symmetry.

**166. Problem.** — *To construct a triangle, having given two of its sides and the angle opposite to one of them.*

There are *two cases* to this problem when the given sides are unequal, according as the given angle is opposite to the greater or the lesser of the given sides. They can best be compared by taking the same given parts in both, as in Figs. 75 and 76.

Let $a$ and $b$ be the given sides, and $c$ the given angle opposite to one of them.

*Case 1.* — In Fig. 75, draw any straight line, AB, and at any point, A, upon it, make an angle, $c$, equal to the given angle, $c$. Lay off from A, on the side AC of this angle, AC equal to the *shorter* given side, $a$. Then, with C as a centre, and radius equal to the *longer* given side, $b$, describe an arc cutting, AB in B, the remaining corner of the triangle, which completes the required triangle, ABC.

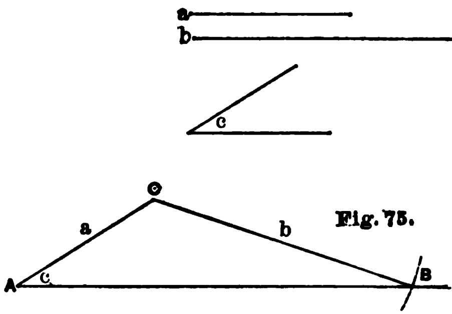

Fig. 75.

*Case 2.* — Draw AB, and make the angle $c$ equal the given angle $c$, as before. But then lay off on AC the *longer* given side, $b$, giving C; and with C as a centre, and the *shorter* given side, $a$, as a radius, describe an arc which will cut AB in *two points on the same side* of A, as at $B_1$ and $B_2$ (read B one and B two), giving *two* triangles, $AB_1C$ and $AB_2C$, each of which contains all the given parts in the manner required; that is, the given angle, $c$, opposite the *shorter* side, $a$.

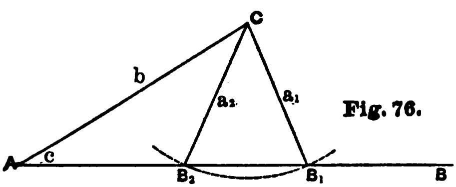

Fig. 76.

It is uncertain in this case, before finding C, whether the given side $a$ will reach to AB. It cannot be less than the perpendicular from C to AB if any solution is possible.

**Practice.** — 1. Let the side $a$ be one inch, $b = 1\frac{3}{4}$ inches, and let $c = 72°$ be opposite to $b$.

2. If, in Fig. 75, the arc with radius $b$ be produced to cut AB again, on the left of A, at a point D, how many and which of the three given parts will the triangle ACD contain?

3. When the arc $B_1B_2$, Fig. 76, just touches AB, how *many* and what *kind* of triangles will be produced?

4. Make the given angle, *c*, obtuse. To which case must the triangle then belong ?

5. What kind of angle must *c* always be in *Case 2 ?*

**167. Problem.** — *To construct a triangle, having given one side and the two adjacent angles.*

Suppose the given side to be 1½ inches, and the angles at the right and left ends of it to be 52° and 69° (Fig. 77).

Draw a line, and lay off on it AB, 1½ inches. At A make the angle A equal to 69°, and at B the angle B equal to 52°. The sides, AC and BC, of these angles can meet nowhere else than at C, because their *starting-points* and *directions* are both fixed.

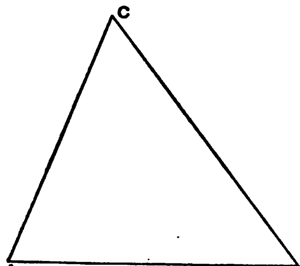

Fig. 77.

Moreover, since the sum of the three angles, A, B, C, is a fixed sum (180°), and the sum of A and B is also fixed, the third angle, C, in this case has a fixed value (141). Hence no other triangle than ABC can be drawn having the same values of the given parts AB, angle A, and angle B.

**Practice.** — 1. In Fig. 77, what is the value of the angle C ?

2. Construct the triangle, having given BC 2 inches, the angle C 77°, and the *exterior* angle at A 107°.

3. Let AB = 1¾ inches, the angle at A = 115°, and at B = 15°.

4. In the last example, what will be the value of angle C ? and why must B be less than 65° ?

**168. Problem.** — *To construct a triangle, having given one side, with one adjacent, and one opposite angle.*

Let the given side AB, Fig. 78, be 1¼ inches, the adjacent angle A = 47°, and the opposite angle C = 59°.

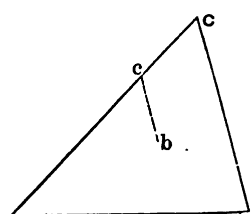

Fig. 78.

We see at once that the angle B = 180° − (47° + 59°) = 74°, which converts this case into the last. But it may be interesting to see that this problem can be constructed directly from the given parts, and to learn, by doing so, that it is a little easier, as well as more natural, to find the value of angle B as above, and then to draw the figure as in the last problem. Then, at A make the angle A = 47°, and draw AC of any length. At any point, *c*, on AC, make an angle, A*cb* = 59°, with the protractor; but instead of drawing the line, *cb*, when the ruler is set at an angle of 59° with AC at *c*, draw BC, parallel to the direction of *bc*, from B, which will give the required triangle.

# CHAPTER V.

## CENTRAL OR REGULAR FIGURES.

### First Principles. Equilateral Triangles.

**169.** In leaving lines and angles and triangles for *plane figures* in general, we are really only leaving the former, *considered apart*, to study them as *combined* to form plane figures.

**170.** *First* in interest and importance, we shall find the *regular* plane figures, called the *regular polygons;* that is, those in which the sides and angles are equal.

No plane figure can be bounded by *less than three straight* lines, for no less number can completely enclose a portion of a plane surface.

No plane figure can be bounded by more than an infinite number of straight lines.

The regular figure bounded by *three equal* straight lines is an *equilateral* (that is, equal-sided) TRIANGLE.

The regular figure bounded by an infinite number of straight lines is no other than a CIRCLE; for, the number of sides being infinite, each is infinitely short, which is the same as saying that the circumference changes its direction constantly, which is true of the circle. Also, for the angles to be equal is the same as to say that the *amount of change of direction* is uniform, which is true only of the circle. Hence the regular polygon of an infinite number of sides is a *circle.*

**171.** The *equilateral triangle* ABD, and the *circle*, Fig. 79, accordingly stand at the two ends of the line of regular polygons, which may have any other number of sides between three and infinity. They are therefore the *extreme cases* of the regular polygons.

**172.** It is a peculiar fact, found by experience, that extreme cases of any kind are useful in the study of other things of the same kind. Thus we see that we could not have studied the previous chapters without mentioning the triangle and the circle, nor could we have put other chapters in their place to avoid mentioning these figures. We can also partly see *why* this is so. As no figure can have less than three sides, all figures of more sides can be divided into triangles. Also, whenever we wish to find or place points somewhere at a given distance from given points, we know that the former points will be on circular circumferences, having the latter points as centres. Thus we can see that the triangle and the circle are of great use in studying all other figures by parts.

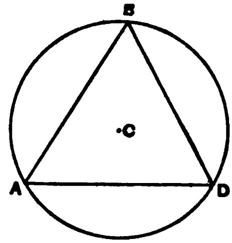

Fig. 79.

**173. Theorem.** — The three angles of an equilateral triangle are equal.

Taking the angles A and B of the equilateral triangle ABC, Fig. 80, AB and AC are equal to BA and BC respectively; that is, B and C are at the same distances from A that A and C are from B. Hence AC has the same position relative to the angle B that BC has relative to A. Hence these lines may measure those angles so far as to show whether they are equal or not. But AC = BC, hence the angles B and A are equal.

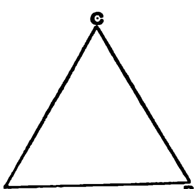

Fig. 80.

In the same manner, it can be shown that A and B each equal C.

**174.** When one or more simple consequences follow so directly from a theorem that the mind readily thinks of them as soon as the theorem is understood, such consequences are called *corollaries.*

Two corollaries follow from the present theorem: —

*First.* — Similar reasoning would apply to the angles at the *base* (160) of an isosceles triangle, which are therefore equal.

*Second.* — If the angles of an equilateral triangle are equal, and if, also (138), their sum is two right angles, or 180°, each one of them is an angle of 60°.

**175.** The following is another proof of the last theorem: —

Suppose a paper equilateral triangle, with sides equal to those of ABC, Fig. 80, and lettered *abc* in the same order as in that figure. Then place *ab* of the new triangle on the equal side AC of Fig. 80. If, then, *bc* should fall anywhere else than on AB, the remaining side, *ac*, would be either longer or shorter than BC, and the triangle *abc* would not then be equilateral. But *abc* is equilateral, with sides equal to those of ABC; and hence, when any side of *abc* is placed on any side of ABC, the other sides of *abc* will fall on those of ABC. But this could only happen by the angles of *abc* being equal to those of ABC.

**176.** Since the sides and angles of an equilateral triangle are equal, its three altitudes, of which CD, Fig. 81, is one, are also equal. For each one of them is a perpendicular to a line of fixed length, the same for all; and from a point at a given distance, also the same for all, from the extremities of that line, this line being in each case one of the equal sides of the triangle.

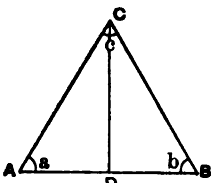

Fig. 81.

**177.** 60° being an exact divisor of 360°, it follows that equilateral triangles can be combined together to form pavements. Also, as 3 × 60° = 180°, or all the angular space around one point and on one side of a straight line, it follows that they can be combined independently on opposite sides of a straight line, as will easily be evident by trial with a number of equal card-board equilateral triangles.

**Practice.**—1. Of what are plane figures composed?

2. When both the lines and the angles are equal in a plane figure, what is the figure called?

3. What is the least number of straight sides that a plane figure can have? and what, then, is the figure?

4. What figure has an infinite number of sides? How often and in what manner does its circumference change its direction?

5. How are the triangle and circle useful in the study of other figures?

6. In what two ways may a person, being at one corner of a plane figure, turn entirely around?

7. In Fig. 81, draw a perpendicular to BC from A, and then prove that the sum of the three angles of ABC are equal to two right angles.

8. Supposing Fig. 81 to be on the ground, walk from A to B, facing B; then turn the back towards C, and step backwards to C; then turn so as to face A, and walk to A; then turn the back to B. Through what angles have you turned, and how far round? and what does that show as to the sum of *a*, *b*, and *c*?

9. In Fig. 81, start at A, and walk around the triangle by way of C and B, instead of by way of B and C, and see if it makes any difference in the truth proved in (174).

10. Having two angles, lay off one inch from the vertex on each side of each angle. If the lines joining these points in each angle are equal, what does that show?

11. Applying this test to all the angles of an equilateral triangle, what does it show as to their size?

12. What kind of a triangle is shown in the same way to have *two* equal angles?

13. Cut out of paper any one of the angles of an equilateral triangle, and apply it to the other two. What do you find about the angles?

14. Produce AC and CB, Fig. 80, and show that the exterior angles so formed are respectively equal to A + B and to A + C.

15. Draw a figure illustrating Art. 176.

16. Cut out a quantity of equal paper equilateral triangles, and arrange them in a row from right to left, and alternately base up and base down.

17. Arrange six of them around one point (forming a regular hexagon). Make other equal hexagons to the right and left of this, and fill the spaces between them with equilateral triangles.

18. Devise other patterns of equal equilateral triangles.

## THE CIRCLE.

### The General Relations of Circles to Straight Lines and to Each Other.

**178.** The line AB, Fig. 82, is wholly *exterior* to the circle of centre, O, and radius, OT. Moving the line nearer the circle, it at last comes to *touch* it at one point, T. Then it is the *tangent*, CD.

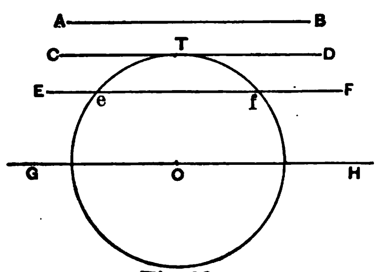


Fig. 82.

Moving the line farther still, it *cuts*, or *intersects*, the circle at two points, and is then called a *secant*. As it approaches the position GH through the centre, O, it is evident, by inspection (129) alone, that the nearer to O the secant is, the *longer* will be the portion, *ef*, which is included within the circle.

**179.** The relative positions of two circles make a good geometrical example by which to cultivate *observation*, in noting them *all;* and *order*, in arranging them in some simple systematic manner.

In Fig. 83, two circles are wholly separate, and exterior to each other. C*c*, the distance between their centres, is made up of their radii, R and *r*, and the distance, *a*, between their circumferences; that is,

$$Cc = R + r + a.$$

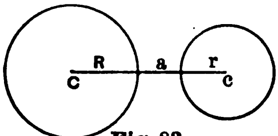


Fig. 83.

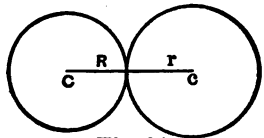


Fig. 84.

In Fig. 84, two circles *touch* one another exteriorly, and *a* disappears or is equal to zero. Then

$$Cc = R + r + 0, \text{ or } = R + r.$$

In Fig. 85, the circles *cut*, or intersect, in two points, and

$$Cc = R + r - a.$$

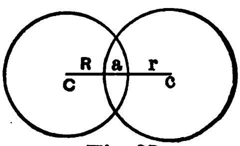


Fig. 85.

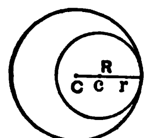


Fig. 86.

In Fig. 86, the circles *touch* again, but *interiorly*, and *a* again disappears. Then

$$Cc = R - r - 0, \text{ or simply } = R - r.$$

In Fig. 87, one circle is wholly within the other without touching it, and

$$Cc = R - r - a.$$

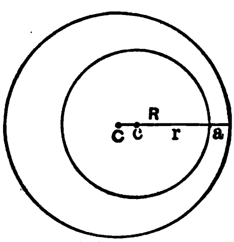


Fig. 87.

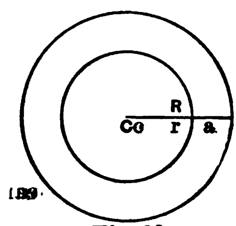


Fig. 88.

Lastly, in Fig. 88, the two circles have the same centre, or are *concentric*, and C*c* disappears, or is equal to zero; that is,

$$Cc = R - r - a = 0.$$

**180.** Assembling the *six equations* which express these *six* positions, we have, —

$$1°.\ Cc = R + r + a.$$
$$2°.\ Cc = R + r + 0 = R + r.$$
$$3°.\ Cc = R + r - a.$$
$$4°.\ Cc = R - r - 0 = R - r.$$
$$5°.\ Cc = R - r - a.$$
$$6°.\ Cc = R - r - a = 0.$$

To *interpret an equation* is to tell what it means; in this case to tell which one of the six figures last given, each one of these six equations means. Compare these equations with the figures until each is immediately understood.

**Practice.** — Let the centres of two circles be $o$ and $o'$, their radii $r$ and $r'$, and the least distance between their circumference $d$. Then, —

1. Draw the pair of circles which are indicated by $oo' = r + r'$, and place the line $oo'$ in a vertical direction.
2. Draw the pair indicated by $oo' = r - r' - d$.
3. Draw the pair indicated by $oo' = r + r' + d$.
4. Draw the pair indicated by $oo' = r + r' - d$.
5. Using the same letters, what will indicate two intersecting circles?
6. What will indicate two concentric circles?
7. What will indicate two circles which are tangent interiorly?
8. What will indicate two that are tangent exteriorly?

### Lines and Parts of Circles.

**181.** So much of a secant as is included within the circle, as *ab*, is a *chord* (Fig. 89).

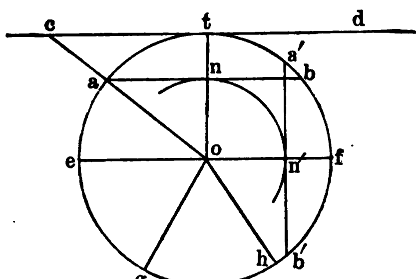

Fig. 89.

The *diameter*, *ef*, is simply the *greatest chord*.

Half a diameter, as $oe = oa = ot = of = og = oh$, is a radius.

An *arc*, as *atb*, terminating on a chord, as *ab*, is said to be *subtended by the chord*. *aghb* is also subtended by *ab*, but the lesser arc is always understood unless it is otherwise stated.

A diameter subtends a semi-circumference.

**182.** *Equal chords subtend equal arcs;* for, owing to the equality of all the radii of any circle, such chords and their arcs may be regarded as only different positions of the same chord and its arc. Thus, suppose *ab* and *atb* and *on* to be immovably fastened together. Then, if *on* turns about *o* to the position *on′*, the chord *ab* will turn to *a′b′*, and the arc *atb* to the position *a′fb′*.

**183.** The radius which is perpendicular to a chord bisects the chord. Thus, *ot*, perpendicular to *ab*, bisects *ab*, or $an = nb$; for $oa = ob$, both being radii of the same circle. Then, by (145), $an = nb$.

The same radius also bisects the arc which the chord subtends. This arises from the uniformity of the circle caused by the equality of all its radii, so that any diameter divides it into halves which will exactly coincide when folded together about that diameter. Thus, *bt*, folded over about *ot*, would exactly coincide with *at*.

**184.** Since equal chords subtend equal arcs, in the same or in equal circles, the greater the chord, the greater the arc; and the less the chord, the less the arc. The two, a chord and its arc, do not, however, increase at the same rate. This is evident on inspection; for a chord very near to a diameter would be very nearly equal to the diameter, because the circumference is there very nearly perpendicular to the diameter, while its arc would be considerably less than the semi-circumference, for the same reason.

But the nearer a chord is to the parallel tangent, the more nearly equal is the increase of the chord and its arc.

**185.** *The radius at the point of contact of a tangent is perpendicular to the tangent.*

For the chord *ab*, Fig. 89, contains two points, *a* and *b*, of the circle, and the point, *n*, of the radius perpendicular to the chord. Now, as *ab* moves, keeping parallel to the position *ab*, and *n* keeping on *ot*, *a*, *n*, and *b* continually approach each other; and, finally, by reason of the continual equality of *at* and *bt* for the different positions of *ab*, they all unite in the one point, *t*, where the moving chord, *ab*, becomes the tangent, *cd*, parallel to *ab*. The radius *ot* is, therefore (131), perpendicular to *cd*.

**Practice.**—1. Draw a circle, and in it three chords, no two of the same length or direction.

2. Draw two *unequal* circles and two *equal* chords, one in each circle.

3. In (2) draw the radii to the extremities of both chords, and note in which circle these radii include the larger angle.

4. Draw a circle and any three radii of it, and prolong them beyond the circumference. Then, where these radii meet the circumferences, draw perpendiculars to them by means of the triangles.

5. In the last problem, construct the perpendiculars by the compasses.

6. In the last two examples, what are the perpendiculars with respect to the circles?

7. In any circle, draw three equal chords and the tangents which are parallel to them.

**186.** Proceeding to the parts of a circle, there are, *first*, the *sector*, which is the part included between any two radii; *second*, the *segment*, which is the part included between a chord and the arc subtended by it; *third*, a semicircle, which is the segment whose chord is a diameter. Also, a sector whose radii are at right angles to each other is called a quadrant, and a semicircle is a sector as well as a segment. It is one whose radii include an angle of 180°, and hence form a diameter.

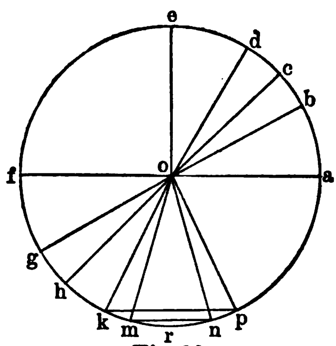

Fig. 90.

**Practice.**—1. In Fig. 90, arc *ab* or angle *aOb* = 30°, and *bOc* = 15°. Then how many times larger is the *sector aOb* than *bOc*?

2. *aOc* is how many times *bOc*? how many times *aOb*?

3. The angle *aOd* = 60°. Then the *sector aOd* is how many times sector *aOb*? *bOc*?

4. *aOe* being 90°, what is the sector *aOe* called?

5. Sector *aOe* is how many times sector *aOb*? *aOc*? *cOd*? *aOd*?

6. *mn* and *kp* are parallel *chords*. Which subtends the greater *arc*? Which bounds the greater *segment*? Which corresponds to the greater *angle at the centre*? Which marks the greater *sector*?

7. The chord which joins the extremities of the radii which include a a sector, divides the sector into what parts?

8. In the same or equal circles, what *five parts* increase or diminish together?

**187.** *The theorem* (138) that the three angles of a triangle are equal to two right angles, is a great help in finding angles of various different numbers of degrees, without any instruments whatever.

Thus, draw a circle on stiff paper by means of a silver dollar or half-dollar, or a round tin or wood box-cover from two to ten inches across (if compasses are not at hand). Cut out the circle carefully, and fold it exactly in halves. The crease where it is folded will be a diameter of the circle. Fold it again so that the two ends of the diameter you have found shall fall together. This will give you a second diameter, at right angles to the first, and their intersection will be the centre of the circle. Now, to be sure that the work is exact, fold the circle again carefully in halves about the first diameter, and then, without unfolding, fold it again about the second diameter, which should give you the circle folded into four exactly equal quarters.

**188.** Let OAB, Fig. 91, represent one of these quarters, each one of which includes a right angle, AOB; that is, 90°. Now fold this *quadrant*, OAB, so that A shall fall on B. It will then be folded in halves on a line, Oc, which will give two angles, as AOc of 45°, or half a right angle, each.

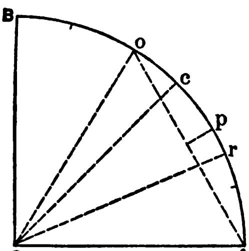

Fig.91.

Next fold over AO until O falls on the arc AB, as at *o*. AO*o* will be two-thirds of a right angle; for, by the folding, the straight line A*o* = AO and AO = O*o* are equal, being radii of the same circle. Then all the sides of the triangle AO*o* are equal. Hence all its angles are equal, and hence each, being *one-third of two* right angles, is *two-thirds of one* right angle, or 60°.

In the same way, fold over BO till O falls at *p*, and we shall have the right angle divided into thirds.

The difference, *oc* or *pc*, between a half and a third is one-sixth; hence *o*Oc is an angle of one-sixth of a right angle, or 15°.

By folding the half right angle, AOc, of 45°, in halves, we shall get a *quarter* of a right angle, AO*r*, or 22½°.

The difference, *pr*, between A*p*, 30°, and A*r*, 22½°, is 7½°, or one-twelfth of 90°, and measures an angle, *p*O*r*, of 7½°.

Finally, the 15° spaces, as *oc*, can be divided quite accurately by the eye into three equal parts 5° each, measuring angles of the same amount.

Thus, by very simple operations, we have found *arcs* and *angles* of

| | |
|---|---|
| 180° = 2 right angles. | 22½° = ¼ right angle. |
| 90° = 1 right angle. | 15° = ⅙ right angle. |
| 60° = ⅔ right angle. | 7½° = $\frac{1}{12}$ right angle. |
| 45° = ½ right angle. | 5° = $\frac{1}{18}$ right angle. |
| 30° = ⅓ right angle. | |

**189.** By performing the operations just described, very carefully on a circle of smooth strong paper a foot in diameter, and with fine lines, the arc of $7\frac{1}{2}°$ can be divided pretty accurately by the eye into three equal parts of $2\frac{1}{2}°$ each, two of which will be a measure to test the 5° divisions, which can then be divided by the eye into five equal parts of 1° each. Thus some idea will be gained of the niceness of the operation of making graduated—that is, equally divided—circles.

**Practice.**—1. Find the centre of a paper circle by folding it (how many times?).

2. Fold the circle into quarters.

3. Fold a quarter-circle so as to show an angle of 45°.

4. Cut a circle in halves, and fold one of its quarters so as to cut off 45°.

5. Cut off the angle of 45°, and the remainder of the half-circle will contain an angle of how many degrees?

6. Fold this remainder in halves, and what angle will you get?

7. Fold a quadrant so as to cut off an angle of 60°.

8. The remainder of the quadrant will be how many degrees?

9. Cut off the angle of 30°, and lay it against the angle of 45° on a quarter-circle. How many degrees in the *sum* of the two angles?

An angle is said to be inscribed in a circle when its vertex is on the circumference of the circle.

**190. Theorem.**—*An inscribed angle is measured by half the arc included between its sides.*

Let *cad*, Fig. 92, be the given angle; the centre, *b*, of the circle being on one of its sides.

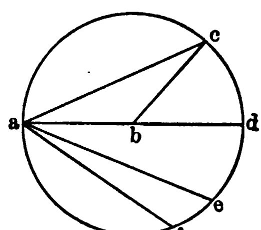

Fig. 92.

By (139), the angle *cbd* is equal to the sum of the angles at *a* and *c*. But *abc* is an isosceles triangle, because its sides, *ab* and *bc*, being radii of the same circle, are equal. Hence the angles at *a* and *c* are equal, and each is therefore half of *cbd*. Now, *cbd* is measured by the arc *cd*. Hence, *cad*, being half of *cbd*, is measured by half the arc *cd*.

*Remark.*—By observing that *cae* is the *sum* of *cad* and *ead*, and that *eaf* is the *difference* between *fad* and *ead*, we can easily prove in a similar manner that *cae* is measured by half the arc *ce*, and *fae* by half of *ef*.

**191.** When, as may often be the case, a person wishes to draw a circle, and has no compasses, it can be drawn, like Fig. 93, with a twenty-five cent piece, or with any other coin or smooth round piece. But it may then be necessary or desirable to know the centre of the circle; and the question therefore arises, How shall that point be found?

The fact that the bisecting perpendicular to any chord contains the centre, enables us to find the centre easily, as shown in the figure. Thus we have the

**Problem.** — *To find the centre of a given circle or arc.*

1°. Draw any two chords, not parallel, as *ab* and *cd*, Fig. 93.

2°. With *a* and *b*, taken in succession, as centres, and any convenient radius, — the same for both, — describe short arcs intersecting each other, as at *e* and *f*. Then *ef* will be perpendicular to *ab* at its middle point, and will therefore contain the desired centre of the given circle (125).

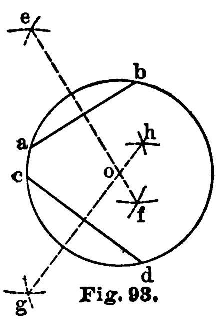

Fig. 93.

3°. Likewise, with *c* and *d* as centres, describe arcs intersecting, as at *g* and *h*, and the line *gh* will, in like manner, contain the centre sought. Hence this centre, being both on *ef* and on *gh*, can only be at *o*, their intersection. Then *o* is the centre of the circle *abdc*.

**192.** The bisecting perpendiculars to any number of other chords of the circle, Fig. 93, would also pass through *o*: first, because a circle can have but one centre; and second, because all these chords would be taken on the same circle.

But attentive examination of Fig. 93 raises the question, If only certain points of a circle are given, how *many* can be taken at pleasure; that is, having no connection with each other, and so that only one circle can be drawn through them?

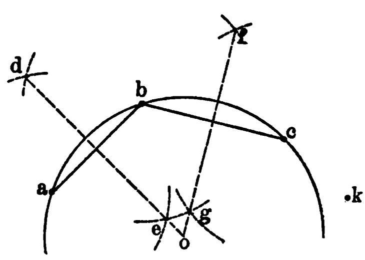

Fig. 94.

*Two points will not be enough;* for, if *a* and *b*, Fig. 94, be two given points through which to draw a circle, the straight line *ab* will be a chord of the required circle, and the centre of the *one* circle desired will be *somewhere* on the perpendicular, *de*, to that chord at its middle point: but we shall not know exactly where, because *any* circle with its centre on *de* will pass through *a* and *b*.

*Three points will be enough;* for, if a circle be required to pass through *a*, *b*, and *c*, the perpendicular to the second chord, *bc*, will also contain the centre of the circle, and the centre will therefore be at the intersection, *o*, of these two perpendiculars, *de* and *fg*; so that, if we take *o* as a centre, and *oa* as the radius of a circle, we shall find that this circle will pass through *a*, *b*, and *c*. Also, *only one* circle can pass through *a*, *b*, and *c*; for two straight lines, as *de* and *fg*, can only meet in *one* point, giving *one* centre, *o*, *one* radius, $oa = ob = oc$, and hence only *one* circle.

*More than three points will be too many;* for if, in Fig. 94, we take *four* points, *a*, *b*, *c*, and *k*, before finding the circle, we shall find that the bisecting perpendicular of the chord *ck* will not meet *de* and *fg* at the same point where these meet each other, unless, by chance, *k* happens to be on the same circle with *a*, *b*, and *c*.

We have thus proved the following: —

**Theorem.** — *Through three points, not in one straight line, one, and only one, circle can be drawn.*

This is what is meant by the shorter form of words, "A circle is *determined* by three given points."

**Practice.** — 1. Draw five different circles through *one* given point.

2. Draw three different circles through *two* given points.

3. What *limit* or *confining condition* is there to the position of the *centre* in the last example?

4. Is there any such limit in the first example?

5. Draw a circle through these three given points: *a* . . *b* . *c*

6. Draw a circle through these three given points: *a* . . *b* . *c*

7. If the chords *ab* and *bc* are of the *same length* in the last two examples, what effect has the angle *abc* between them (obtuse in (5) and acute in (6)) on the size of the circle containing the given points?

**193.** Having now considered the two extreme regular polygons, the equilateral triangle, and the circle, we will take up such other regular polygons as are most important, beginning with

## THE SQUARE.

*The square*, Fig. 95, is a figure of *four equal sides and four equal angles*. These are *right angles*, because a person, starting at A and facing B, and walking round the square to the starting-point, would have turned entirely around once; and because, in turning equally at four different points, B, C, D, and A, he would have turned a quarter round, or one right angle, at each point.

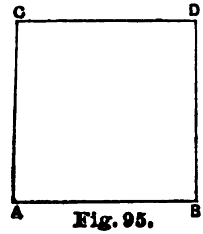

Fig. 95.

**194.** If we pin together *three* thin strips of wood or card-board so as to form a triangle, as in Fig. 96, it will be found that they will form a *rigid* or *inflexible* frame; that is, one whose *shape cannot be altered*, or, in still other words, a *figure of invariable form*. But if we pin together *four* equal strips to form a square, we shall find that the corners or *joints* will be flexible; that is, a square is not a rigid figure, and the same is true of all figures, except the triangle.

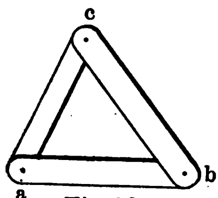

Fig. 96.

**195.** The lines, as AD and BC, Fig. 97, which join opposite corners of a square are called its *diagonals*.

The square has two diagonals.

Each diagonal divides the square into two triangles, as ABC and BCD, which are evidently, 1st, equal; 2d, right-angled; 3d, isosceles. Also, each of the two angles at A, B, C, and D, being evidently half a right angle, is an angle of 45°.

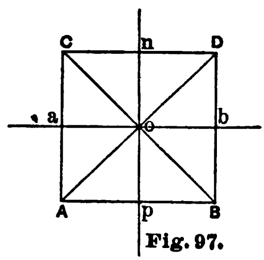

Fig. 97.

**196.** By examining the square in connection with the idea of *symmetry* (143), we see that it has *four axes of symmetry*, or centre lines, in *two pairs:* one pair, AD and BC, containing opposite corners; the other, *ab* and *np*, the middle points, as *n* and *p* of opposite sides; and all containing the centre, *o*, of the square.

Further, we see that the two diagonals divide the square into four equal triangles, each of which, as A*o*B, is therefore one-quarter of the square, and that the four centre lines divide the square into eight triangles, as A$op$, all of the same shape as the half-square ABD, and each equal to one-eighth of the square. The eight angles at $o$ being thus equal, each is 45°.

**197.** Since $ao = bo = po = no$, a circle of centre, $o$, and radius, $oa$, would be tangent to the sides of the square at $a$, $p$, $b$, and $n$. This is called an *inscribed circle.*

Since, also, A$o$ = B$o$ = C$o$ = D$o$, a circle of centre, $o$, and radius, A$o$, would contain the four corners of the square, and be its *circumscribed circle.*

Hence the point $o$ is properly called the *centre* of the square, it being as nearly as possible the same thing for the square that the centre of a circle is of the circle, as we will soon show more fully.

**Practice.**—1. How many sides has a square?

2. How do their lengths compare?

3. How many angles has a square, and of what size?

4. Does the *position* of a square affect its *form?*

5. Can different squares be of different *shape?*

6. How can they differ?

7. What difference in action is there between a three-sided and a four-sided plank frame pinned at the corners?

8. How many pieces, and how placed, would make the square frame of unalterable shape?

9. What and how many are the diagonals of a square?

10. How does a diagonal divide the square?

11. In what ways can a square be divided into halves?

12. How many different shapes will these halves have?

13. What is the centre of a square?

14. How many different *sizes* of triangles can be formed out of a square, 1st, by one diagonal; 2d, by both; 3d, by adding to these all its centre lines?

15. How do these triangles compare in *form?*

16. What is their form?

17. Draw a square with its inscribed and circumscribed circles.

### Drawing of a Square.

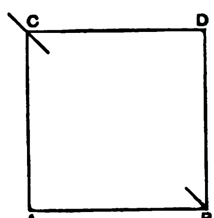

Fig. 98.

**198.** *To draw a square on a given side,* AB, Fig. 98.

*First,* without using the compasses.

At A draw a perpendicular, AC, to AB by means of the ruler and triangle, as in (74); limit AC at C by the diagonal from B. This may be done by making the angle ABC one of 45° by a protractor, or by placing the 45° triangle (69) against a ruler placed on a parallel to AB. Then draw CD parallel to AB, and BD perpendicular to AB, by means of the ruler and triangle.

**199.** *Second*, by means of the compasses, Fig. 99. Draw the perpendicular at A, either as in (74) or as in (125). Then limit it by a short arc with centre A and radius AB. Lastly, find D by means of arcs having C and B as centres, and a radius equal to AB.

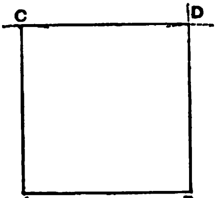

Fig. 99.

**Practice.**—1. On a given side of 2 inches, draw a square by the triangles.

2. On a given side $1\frac{1}{2}$ inches long, construct a square by the compasses.

3. Draw a square with the 45° triangle on a given diagonal.

## INSCRIPTION AND CIRCUMSCRIPTION.

### The Square and the Circle.

**200.** Any polygon all of whose corners, angles, or vertices lie in the circumference of a circle, as in Fig. 100, is said to be *inscribed in* the circle. Thus the square ABCD is inscribed in the circle whose radius is OA (197).

Any polygon all of whose sides are tangent to the circumference of a circle, is said to be *circumscribed about* the circle. Thus the square EFGH is circumscribed about the circle ABCD, Fig. 100.

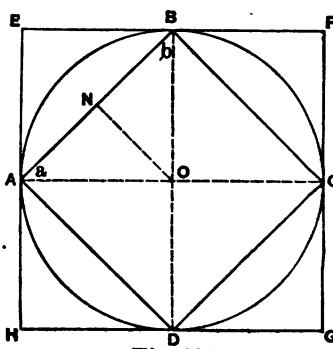

Fig. 100.

**201.** In the two last statements we have mentioned the *square in relation to the circle.* But we can think of the square first, and so think of the *circle in relation to the square.* Thus we can say that the circle of radius OA is inscribed in the square EFGH, and that it is circumscribed about the square ABCD.

**202.** Since all the angles of a square are equal, being right angles, and are also directly opposite each other on account of the equality of the sides, the *diagonals*, AC and BD, bisect these angles, giving *four isosceles triangles* which are *equal*, because there is a *side* and *two angles*, as AB and its adjacent angles *a* and *b*, equal in all of them. Hence OA, OB, OC, and OD are all equal; that is, *the circle and its inscribed square have the same centre* (197).

**203.** Again: but, as the diagonals AC and BD, which, as just seen, bisect each other, are equal and parallel to the sides of the circumscribed square EFGH, they evidently divide it into *four equal squares*, having a common vertex at O.

The centre, O, of the circle OA is therefore also the centre of its circumscribed square.

**204.** What we have now shown to be true of the square, we shall see plainly enough, from other figures, to be true of *all regular polygons;* that is, —

1°. The bisecting lines of the angles of every regular polygon divide it into as many *equal isosceles* triangles as the polygon has sides; and their common vertex is equidistant both from the sides and the angles of the polygon, and is the centre of the polygon.

2°. The centre of a circle is also the centre of all its inscribed and circumscribed regular polygons, whatever may be the number of their sides. This principle will be found very useful in the construction of polygons.

**Practice** (see Figs. 97 and 100). — 1. Inscribe a square in a given circle.

2. Inscribe a circle in a given square.

3. Circumscribe a given circle by a square.

4. Circumscribe a given square by a circle.

5. Having a circle and inscribed square, circumscribe a square about the same circle, with its sides parallel to those of the inscribed square.

6. Inscribe a square in a given square so that the angles of the inner square shall bisect the sides of the outer ones (Fig. 100, omitting the circle).

7. Draw the diagonals of *both* the squares described in the last example, and note the *number* and *kind* and *relative size* of the triangles formed by the sides and diagonals of both squares.

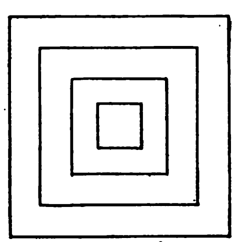

**Fig. 101.**

**205.** *All squares are similar figures;* for a square is composed of four sides and four angles, and of nothing else. But all of these are fixed in the definition of a square, — the *sides* because they must all be equal, and the *angles* in the same way. Hence there is nothing which can be changed so as to produce a change of shape. Hence all squares are of the same shape; that is, they are similar, as is shown in Fig. 101.

**Practice.—1. Draw a four-sided figure with only three of the sides equal, and see what form will be produced.**

**2. Draw a four-angled figure, and see how many right angles it can have, if *all* are not right angles.**

### Figures nearly related to the Square.

**206.** A *regular* figure in geometry means, as said before (170), one in which both all the sides and all the angles are equal.

But there are other figures which in this sense are only partly regular, but yet are regular in the usual sense of the word, and are closely connected with the fully regular ones, both in thought and in the practical arts.

Those belonging with the square we therefore give here.

**207.** If a square, ABCD, Fig. 102, in the surface of the paper be turned a certain amount about the centre line, OX, as described in (160), it will appear narrower, as a blind or door partly open looks narrower, being seen obliquely, than it does when shut. Thus the square may be turned so as to appear as at *abcd*, which has four right angles, *a*, *b*, *c*, and *d*; but only the opposite sides are equal, $ab = cd$, and $ad = bc$. This figure is called a *rectangle*. It is even oftener seen than a square.

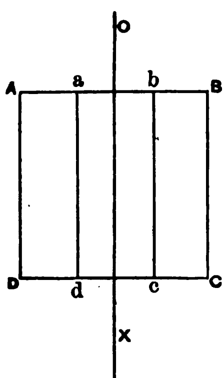

Fig. 102.

The faces of a brick, doors, windows, panes of glass, books, floors, tables, house-lots, and many other things, are oftener rectangles than squares.

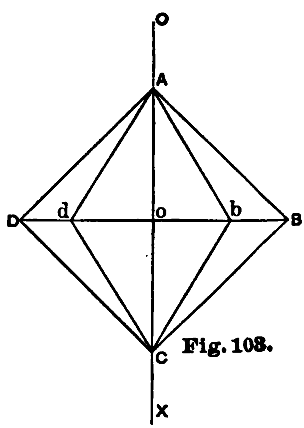

Fig. 103.

**208.** Again, if the square ABCD be set in the position shown in Fig. 103, and then revolved a certain distance about the axis OX, or until B revolves *forward*, for example, to *b* (in *front* of the paper), and D an equal amount *backward* to *d* (an equal distance *behind* the paper), the square will appear of the form AC*bd*, called a *rhombus*.

Thus a rhombus is a figure of *four equal sides*, in *parallel pairs*, and whose *pairs of opposite angles are equal.* Angle A = angle C, and angle *b* = angle *d*.

**209.** But the rhombus may be derived from the square in another way, illustrated by Fig. 104. Remembering that a jointed square is flexible (194), if ABCD is a jointed square, and if B be pressed inward as far as *b*, the other three joints, A, C, and D, yielding, and turning on their pins, will take the positions *a*, *c*, and *d*; found by making *ba* = BA, *bc* = BC, and D*d* = B*b*, or, which amounts to the same thing, B*d* = D*b*.

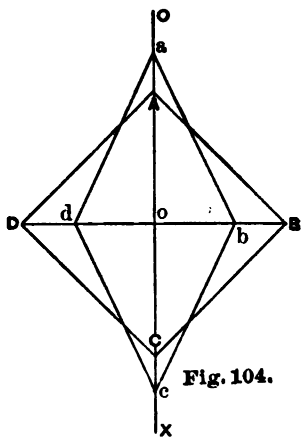

Fig. 104.

**210.** By inspection of Figs. 103 and 104, we see, *first*, that the diagonals of a rhombus bisect each other at their intersection, *o*; also, *second*, that each of them divides the rhombus into two equal isosceles triangles, as C*d*A = C*b*A, or A*bd* = C*bd*; and, *third*, that the two divide the rhombus into *four equal* right triangles, A*o*B = C*ob*, etc. These simple facts, together with the equality of the sides, enable one easily to draw a rhombus.

A rectangle whose adjacent sides are given, can be constructed very nearly in the same manner as the square was in Fig. 99.

**Practice.**—1. Construct a rectangle whose adjacent sides are 1¼ inches and ¾ inch.

2. Construct a rhombus, having given its diagonals, 2 inches and 1 inch.

3. Construct the rhombus of which a side is 1½ inches, and whose larger angles are each 110°.

4. Construct the rhombus whose side is 1¾ inches, and whose smaller angles are each 60°.

### The Square and the Right Triangle.

**211.** There is a remarkable relation between these figures, which is also often and highly useful. It is as follows:—

**Theorem.**—*The square which has the hypothenuse* (142) *of a right-angled triangle for one of its sides is equal to the sum of the squares which have the other sides of the triangle for their sides.*

Let ABC, Fig. 105, be a right triangle. P is the square on AB; Q is the square on BC. Then, PQ, the square on the hypothenuse AC, is equal to the sum of the squares, P and Q, on the other two sides.

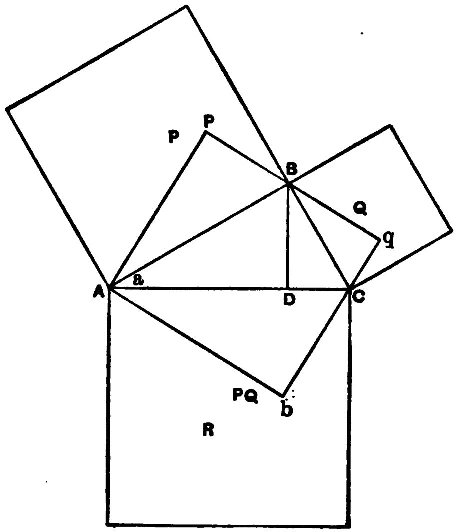

Fig. 105.

Drawing the perpendicular, BD, to AC, we know, 1st, by (155), that ABC, ABD, and BCD are similar triangles.

Also, 2d, all squares are similar figures (205); hence the squares P, Q, and R are similar figures.

But, 3d, the similar triangles have *similar positions* in the squares, the hypothenuse of each triangle being the side of the square which contains it.

Hence, 4th, as may be more evident to the eye by revolving each triangle over into its own square, as at A*p*B, B*q*C, and A*b*C, the triangles are like parts of their respective squares.

Hence, 5th, if the triangle A*b*C = A*p*B + B*q*C, a like relation will exist between the corresponding squares, and

$$\text{Square R} = \text{square P} + \text{square Q},$$

which is written

$$AC^2 = AB^2 + BC^2.$$

Thence, knowing that the sign ($\sqrt{\ }$) means the square root, we have

$$AC = \sqrt{AB^2 + BC^2},$$

or

$$AB = \sqrt{AC^2 - BC^2},$$

and

$$BC = \sqrt{AC^2 - AB^2}.$$

**212.** The above theorem is but one of many similar ones; for the same reasoning shows that *any figure* standing on AC is equal to the sum of the *similar figures similarly placed* on AB and BC. Thus, for example, a circle having AC for its diameter, equals the sum of the circles on AB and BC as diameters.

This famous ancient theorem * has a multitude of useful practical applications, of which the following are a few examples. In solving them, draw a figure showing the given parts.

**Practice.**—1. The other sides of a right triangle are 3 inches and 4 inches. Find the hypothenuse.

2. The hypothenuse is 36 feet; one of the other sides is 24 feet. Find the third side.

3. An attic window is 25 feet from the ground. How long must a ladder, set eight feet out from the house, be to reach the window?

4. A roof is 32 feet wide at the base, and its ridge 10½ feet above its base. What is the length of its slope?

5. Two roads cross at right angles. If two horses, starting together at the crossing, travel, one on each road, how far apart will they be when each is 50 miles from the crossing?

6. A right-angled common has two sides 800 feet long, and two sides 500 feet long. How long is the path from the middle of one side to the middle of the next?

7. The diagonals of a garden-bed, in the form of a rhombus, are 15 feet and 8½ feet. How many *yards* is it round the bed?

8. Find the altitude of an equilateral triangle whose side is 18 inches.

9. A triangular city lot is divided by one of its altitudes, which is 60 feet, into two right triangles. The hypothenuse of one of them is 75 feet, and the base of the other is 112 feet. What is the sum of the three sides of the lot?

10. One guy-rope of a derrick is 96 feet long, and is fastened 72 feet from the foot of the derrick. How high is the post of the derrick?

11. At what distance from the foot of a wall 30 feet high must a timber 40 feet long be placed if the upper end of the timber is to be 3 feet below the top of the wall.

12. If one floor contains 24 square yards, and another floor 18 square yards, how many feet will there be in the side of a square floor equal to both of the given floors?

13. A flagstaff 40 feet high blew over against a building 25 feet from it. At what height did the pole strike the building?

14. How many steps of 2½ feet each, and how many feet, shall I save by going from one corner to the opposite corner of a rectangular field 300 feet by 450 feet, instead of going by the road around two sides of it?

---

* Discovered by Pythagoras, a Greek philosopher, born about 570 B.C.

## THE REGULAR PENTAGON.

**213.** Next in order to the square comes the regular pentagon, or figure of five equal sides and angles. But the pentagon is of comparatively small value for practical uses, partly because it is less easily constructed than the other common polygons, and partly because pentagons do not combine to form patterns as well as other polygons do. We will therefore briefly study its principal features, and show sufficiently how to construct it.

Fig. 106 represents a regular pentagon. As its sides are five equal chords of the circle containing it, each of them subtends

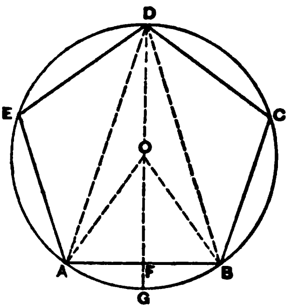

Fig. 106.

an *angle*, as AOB $= \frac{360°}{5} = 72°$, at the centre of the figure. But each *triangle* like AOB is isosceles; hence each angle as OAB $= \frac{180° - 72°}{2} = 54°$, or EAB, etc., each $= 108°$.

Again, ADB $= \frac{1}{2}$ AOB (190) $= 36°$. Then ABD = BAD $= \frac{180° - 36°}{2} = 72°$; or, as ADF $= \frac{1}{2}\ 36°$, or $18°$, and AFD $= 90°$, we may say that BAD $= 180° - (90° + 18°) = 72°$. Also, AOF $= 36°$.

Again, AOD is an isosceles triangle; hence OAD is an angle of $18°$, as well as ADO. Hence AOD $= 180° - 36° = 144°$, which is also evident from its being measured by the arc, AED, of $2 \times 72° = 144°$.

Finally, AED and BCD are equal isosceles triangles, whose vertical angles at E and C are each $108°$, and their base angles are $36°$.

**214.** We thus learn from the pentagon the angles $18°$, $36°$, $54°$, $72°$, $108°$, $144°$, all of which, as will be seen, are divisible by 18.

Moreover, $\frac{18}{36} = \frac{1}{2}$, $\frac{18}{54} = \frac{1}{3}$, $\frac{18}{72} = \frac{1}{4}$, $\frac{18}{108} = \frac{1}{6}$, $\frac{18}{144} = \frac{1}{8}$, $\frac{36}{54} = \frac{2}{3}$, $\frac{54}{72} = \frac{3}{4}$, $\frac{54}{144} = \frac{3}{8}$, all of which are very simple values, showing that all these angles have simple *relations of size;* that is, simple *ratios* to each other. The triangles which they form are therefore of beautiful form, from the flattest one, AOD, to the most pointed one, ADB, for flat or steep roof, window, or other ornamental gables.

**215.** But as 108°, the angle AED, etc., is not an exact divisor of 360°, pentagons will not combine together to form pavements of tiles, but will leave empty angular spaces of 12° each, $= \frac{360° - 324°}{3}$, wherever three of them are placed together, with corners meeting at one point, as A.

**216.** A pentagon, like any other regular polygon, may be made on a given side, or inscribed in a given circle.

**Problem.** — *To construct a regular pentagon on a given side.*

Let AB, Fig. 107, be the given side. Then, —

*First Method.* — At A and B make angles of 54°, whose sides, AO and BO, will meet at O, the centre of the pentagon and of the circle containing it. Then, with O as a centre, and OA as a radius, describe a circle, on which AB can be laid off as a chord four times more, giving the pentagon ABCDE.

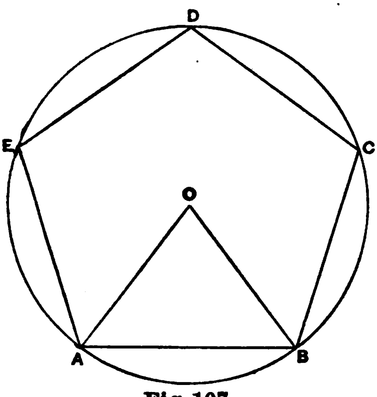

Fig. 107.

*Second Method* (without drawing any circle). — At A and B construct angles, as ABD and BAD, Fig. 106, of 72°. Their sides, AD and BD, will meet at the vertex, D, opposite to AB. Then, on AD and BD as bases, construct isosceles triangles whose angles at A and D shall be 36° each, which will give the remaining vertices, C and E.

**217. Problem.** — *To inscribe a regular pentagon in a given circle.*

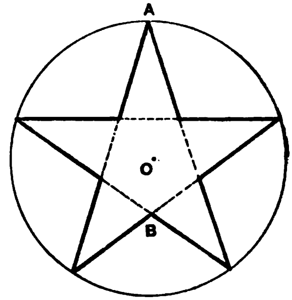

Fig. 108.

Let O be the centre, Fig. 107, and OA the radius of the circle. At O construct two radii including an angle of 72°, and note their intersections, A and B, with the circle. Then the chord AB will apply five times to the circumference, giving the required regular pentagon. After a little practice, the length of AB can be quickly found by trial.

**218.** The beauty of form of a pentagon and its component triangles, depending on the very simple numerical relations between their angles, gives superior beauty to the five-pointed star, derived, as in Fig. 108, from the regular pentagon.

**Practice.**—1. Cut out three equal regular pentagons, and place them together to show how they fail to combine to fill a surface.

2. Sketch one or more door or window openings surmounted with ornamental gables of the form of ABD, Fig. 106.

3. Construct a regular pentagon on a given side two inches long.

4. Inscribe a regular pentagon in a circle of one inch radius.

5. Circumscribe a regular pentagon about a given circle by drawing a tangent *at each vertex* of the inscribed pentagon.

6. Do the same by drawing a tangent *parallel to every side* of the pentagon (or perpendicular, as at G, Fig. 106, to each diameter, as DG).

7. In Fig. 108, draw a line from every external angle, as A, to every opposite internal angle, as B.

## THE REGULAR HEXAGON.

**219.** The regular hexagon, Fig. 109, is bounded by *six equal sides*, which form with each other *six equal angles*.

We *first* notice, that, as the six equal chords, as AB, subtend *six* equal angles at the centre, O, of the hexagon, each one of them, as AOB, $= \frac{360°}{6} = 60°$.

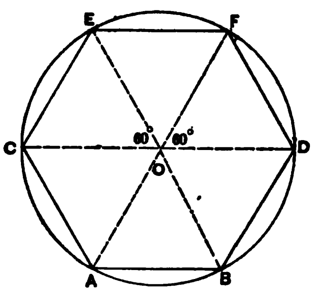

**Fig. 109.**

But, as OA = OB, the angles at A and B are equal, and hence each is 60° (175). Hence the triangle AOB is *equilateral*. But the like can be shown for every side; the hexagon therefore consists of six equal equilateral triangles, and each of its sides, as AB, is thus equal to the radius of the circle which contains the figure.

Hence a regular hexagon requires only the very simple construction of stepping off one of its sides as a chord six times, on the circumference of a circle having the length of a side for its radius.

**Practice.**—Draw a regular hexagon on a given side of three-fourths of an inch.

**220.** We see, in Fig. 109, three equal diagonals, each of which, *first*, is a diameter of the containing circle of the hexagon; and, *second*, is parallel to two opposite sides of the hexagon; and, *third*, divides the hexagon in halves.

*Six equal shorter diagonals* could also be drawn, as from A to D, B to F, D to E, F to C, etc. These will be, *first*, in three parallel pairs. AD and CF, etc.; *second*, each *pair* perpendicular to one of the long diagonals, as AD and CF are perpendicular to BE; and, *third*, each *one* forming with two of the *sides* an isosceles triangle, as ACE, Fig. 110.

**Practice.** — 1. Draw all the short diagonals of a hexagon.

2. What figure do the three which are not parallel to each other form?

3. How many such figures are formed by the six short diagonals, and how are they placed with respect to each other?

4. Draw only so much of each of the equal figures formed by the two sets of three short diagonals as is outside of the other. What ornamental figure will be formed?

**221.** Continuing the study of the diagonals of the hexagon, we see that if one of the long diagonals, as CD, Fig. 109, is horizontal (the hexagon being supposed to be vertical, or drawn on a wall), the other two, AF and BE, will make angles of 60° with it and with each other. Also, two of the *sides*, AB and EF, will be horizontal, and the others will be parallel to the oblique diagonals.

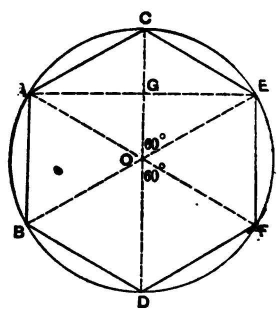

Fig. 110.

But if, as in Fig. 110, one of the diagonals be vertical, as at CD, the others, being inclined to it and to each other at angles of 60°, as before, will be inclined at an angle of 30° with a *horizontal* direction. Also, as two sides are parallel to each of the diagonals, two of the sides will be vertical, AB and EF, and the others will make angles of 30° with a horizontal direction.

By reason of these angles, each shorter diagonal, as AE, Fig. 110, together with the sides nearest to it, as AC and EC, forms an isosceles triangle, ACE, whose base angles at A and E are 30°, and whose vertex, C, is therefore an angle of 120°.

We see, therefore, immediately, that as 120 is an exact divisor of 360, hexagonal tiles can be combined to form pavements, as shown in Fig 111.

**222.** To draw a group of hexagons, very great care is necessary, which, however, affords excellent practice in exactness.

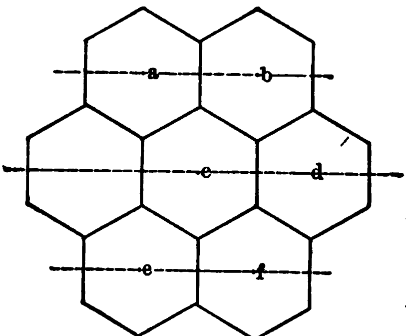


Fig. 111.

1°. Every thing used must be clean and dry.

2°. Pencils must be very hard and sharp, unless the work is done on a board, and made very large.

3°. All *given distances* must be made exactly of the intended length.

4°. All *given points* must be marked very finely, and exactly where they are meant to be.

5°. All lines must be drawn exactly through the points through which they are meant to pass.

**223.** Having made up the mind beforehand to take time and pains, the following is one of several good ways of drawing a pavement of hexagons. *First*, draw, as perfectly as possible, one hexagon of the intended size, as in Fig. 110. Then, *second*, draw as many horizontal (parallel) lines, as *ab*, *cd*, Fig. 111, as there are to be rows of hexagons, and make them at a distance apart exactly equal to DG, Fig. 110; that is, three-fourths of the diameter or long diagonal, CD (220). *Third*, take the centres, *a*, *b*, etc., of the proposed hexagons on these lines, and at a uniform distance apart, equal to a short diagonal, as AE. But, *fourth*, the centres, as *a* and *e*, *b* and *f*, etc., on *alternate* parallels, as *ab* and *ef*, will be directly opposite each other; that is, *ae*, *bf*, etc., will be perpendicular to *ab* and *ef*. The centres, as *c* and *d*, of the other rows will be on perpendiculars to *ab*, just halfway between *ae*, *bf*, etc. *Fifth*, having thus very carefully located the centres of as many hexagons as are desired, draw very fine *equal* circles with these centres, and a common radius equal to that containing the given hexagon. These circles will intersect each other so as to give the corners of the hexagons, which can then be drawn.

**Practice.**—1. Draw twelve equal hexagons in three rows, having a given side of half an inch.

2. Draw (very faintly) *perpendiculars* to *ab*, etc., bisecting *ab*, *cd*, etc. Draw *parallels* to *ab*, *cd*, etc., above and below them, and at a distance from them equal to one side of the hexagon. The intersections of the two sets of lines will give all the vertices of the hexagons. Connect these by heavy lines, observing that the oblique sides make angles of 30° with the horizontal lines.

**224.** Some interesting results flow from the fact that the regular hexagon is composed of six equal equilateral triangles (or *regular trigons*).

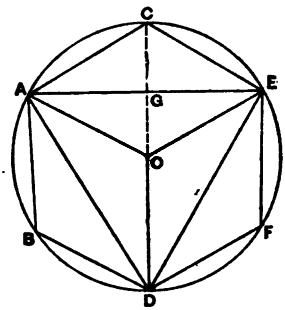

Fig. 112.

*First.* — Since AC = AO = CO = CE = OE, the figure AOEC is a rhombus (208); its diagonals, AE and CO, bisect each other, making GC = GO = ½ OD, so that GD = ¾ CD.

*Second.* — AE = AD = DE, so that ADE is an equilateral triangle.

*Third.* — Drawing the three alternate radii, OA, OE, and OD, heavy lines, the figure, especially if shaded, catches the eye as a picture of a cube showing three faces equally; viz., the top, ACEO, and the sides, AOBD and EODF.

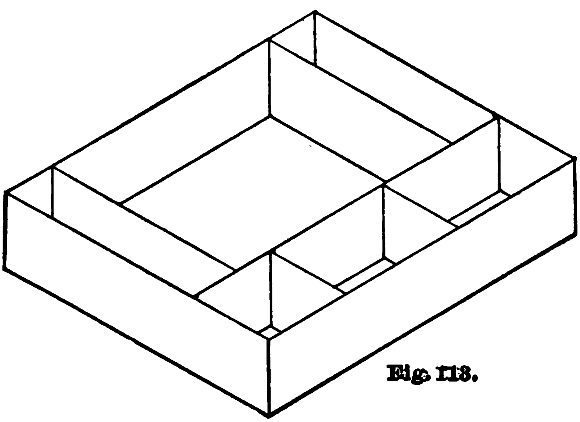

Fig. 113.

**225.** This way of representing a cube is the foundation of the well-known and very useful "isometrical drawing," examples of which have been seen in Figs. 24 and 26. Isometrical means equal measure, the six visible edges and three visible faces of the cube being seen equally. This kind of drawing is most usefully applied to small square-cornered objects. It is quite pretty and entertaining as applied to partitioned box-work and like objects. See Fig. 113, representing a partitioned tin tray, and showing every line on the same scale.

**Practice.** — 1. Make an isometrical drawing of any simple square-cornered table, box, or stool.

2. Make an isometrical drawing of a cube 2 inches square, with a *tablet* 1 inch square, and projecting ¼ inch on each face.

3. Make an isometrical drawing of the same cube, but with a *panel* sunk ⅛ inch into each face.

4. In Fig. 110, draw tangents parallel to every side of the inscribed hexagon, and, by their intersections with three long diagonals, form a regular hexagon, circumscribed about the circle.

5. In Fig. 110, draw tangents at both extremities of each of the long diagonals, and note how, by their intersections, they also form a circumscribed regular hexagon, and that they meet each other on radii perpendicular to the sides of the *inscribed* hexagon.

## THE REGULAR HEPTAGON.

**226.** The regular heptagon, or polygon of seven equal sides, is of comparatively little use; also, there is no simple geometrical construction for it, only the mechanical one of finding by trial a distance which will apply seven times as a chord to the circumference of a circle. But this can easily be done by considering that the side of a heptagon is a little less than that of a hexagon; in fact, is very nearly seven-eighths of the radius of the circumscribing circle.

**Practice.** — Inscribe a heptagon in a circle of 1 inch radius.

**227.** Whenever any polygon is to be constructed on a given side, a larger quantity is derived from a smaller one, and errors are thus liable to be magnified or multiplied as the construction proceeds. Hence, as early in the construction as possible, the containing circle should be found and drawn, which holds all the angles of the figure to their known place. We will illustrate by the heptagon, Fig. 114.

With A and B as centres, and *any* radii (though equal ones equal to AB are most *convenient*, while larger ones give greater accuracy), draw two semicircles, as shown. Divide each of these by trial into seven equal parts, as shown.

Draw AC2 and BD to the *second* points of division. Next, find O, the intersection of the bisecting perpendiculars of the three equal sides CA = AB = BD, which will be the centre of the circle, of radius OA, which will contain the heptagon. We now see that either one only of the semicircles would have been sufficient, but that each is useful as a check on the other in making the location of O perfectly accurate.

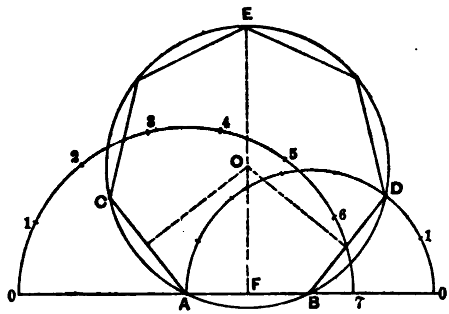

Fig. 114.

Draw the circle OA, and on it lay off AB four times more, which will complete the required heptagon ACEDB.

**Practice.**—Construct a heptagon on a given side 2 inches long.

**228.** The construction just given applies to other polygons by simply dividing the semicircles into as many equal parts as the required polygon has sides.

The heptagon is little used, partly because its angles do not allow it to combine so as to form pavements; also, a polygonal tower, or recess, is more agreeable when built on a plan consisting of a polygon, or half-polygon, of an even number of sides.

**Practice.**—1. Draw the perpendiculars from *all* the vertices of the heptagon to the opposite sides, as from E to AB.

2. How do these perpendiculars divide those sides?

3. Where do they all intersect each other? and why?

4. How many axes of symmetry, or centre lines, has the heptagon?

5. Construct regular polygons of 5, 9, and 11 sides on a given side of $1\frac{1}{4}$ inches, by the method of Fig. 114.

6. Circumscribe a regular heptagon about a given circle.

7. Draw a seven-pointed star. [Draw *very faintly* all the *equal shortest* diagonals, as AD, BC, DE, etc., and all the equal isosceles triangles, like EAB, having the sides of the heptagon for their bases. Then *either* redraw in heavy lines so much of each of these triangles as is *outside of the diagonals*, or so much of each side of each as extends *from a vertex of the heptagon to a side of the next triangle.*]

## THE REGULAR OCTAGON.

**229.** This polygon of eight equal sides and angles is easily inscribed in a circle, as follows (Fig. 115): Having a circle of centre, O, and radius, OA, draw two diameters at right angles to each other, AB and CD, either by compasses or by triangle; bisect each of the four right angles thus formed, either by compasses, as in Fig. 43, or by a triangle, like Fig. 17, having angles of 45°. The points, as E, thus found, will, with A, C, etc., be the vertices of the required regular octagon.

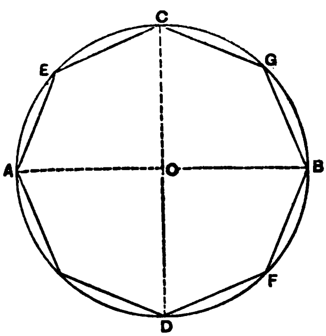

Fig. 115.

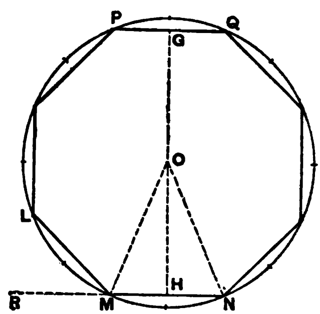

Fig. 116.

**230.** The figure is repeated in Fig. 116 just to show how differently the octagon strikes the eye, according as a vertex, D, Fig. 115, or a side, MN, is put foremost. In the former case the *eight equal angles* are most conspicuous; in the latter case the *four pairs of opposite parallel sides* are the most obvious feature. In Fig. 116, the quadrants, as AC, Fig. 115, are divided into *four* equal parts by bisecting their *halves*, as AE and EC, in order to find P, Q, etc.

Comparing Figs. 115 and 116, we see, from Fig. 115, that there are *four* diagonals joining opposite corners, as CD, each of which evidently is a centre line betwen two opposite and equal halves, as CAD and CBD. Fig. 116 shows that there are four more centre lines, each, like GH, perpendicular to a pair of opposite sides, as MN and PQ, at their middle points, G and H.

**231.** We have now seen a sufficient number of regular polygons, from three to eight sides, to show that each has *as many centre lines as it has sides*, but that, in the polygons of an *odd* number of sides, as the trigon (equilateral triangle), pentagon, and heptagon, these centre lines are *in one set*, each being the perpendicular from a vertex to the middle point of the opposite side, as EF, Fig. 114, and *all equal*, while in the polygons of an *even* number of sides, as the square, hexagon, and octagon, these centre lines are *in two sets*, one joining opposite *corners*, and the other the middle points of opposite *sides*. The latter are shorter than the former.

**Practice.**—1. Inscribe an octagon in a circle of 1 inch radius, and so that a *vertex* shall be foremost.

2. Inscribe an octagon in a circle of 1 inch radius, but, as in Fig. 116, so that a *side* shall be foremost.

3. Draw all the diagonal centre lines of an octagon.

4. Draw all the shorter centre lines of an octagon.

**232.** *Other Diagonals.*—Besides those diagonals which are also centre lines, there are others which are not, since all the lines, except the sides, which join corners are diagonals. A diagonal may be drawn from any corner, as C, to the *second*, B, the *third*, F, or the *fourth*, D, corner from it. The first, CB, is one side of a square, whose corners are C, B, D, and A, and is one of *eight* similar ones. The second, CF, is parallel to a side, GB, and is also one of eight. The third is a centre line.

**Practice.**—1. In an octagon placed as in Fig. 115, join the alternate angles, beginning with C.

2. Join the alternate angles, beginning with E.

3. Join both sets of alternate angles in the same figure, and note and describe the figure formed.

4. Draw all the diagonals which are parallel to the sides of an octagon.

5. How many of those last mentioned start from each vertex?

6. How many parallel pairs of them are there?

7. Having drawn very lightly all the diagonals in example 4, then at every vertex of the octagon, placed as in Fig. 115, draw each of the diagonals, as CF, heavy from the vertex to its intersection with the similar diagonal from the next point, as G, and note what ornamental figure they form.

**233.** ***Angles of the Octagon.*** **—O, Fig. 116, being the centre of the octagon, the angle, MON, at the centre is one-eighth of four right angles, or 45°. Then, as MON is an isosceles triangle, the angles at M and N together are 180° − 45 = 135°, and each of them is therefore ½ of 135°, or 67½°. But as the octagon is plainly composed of eight isosceles triangles, all equal to MON, the angle LMO = OMN = 67½°. Hence LMN = 2 × 67½° = 135°, and therefore the exterior angle LMR = 45°.**

**234. An octagon can easily be constructed on a given side by means of its angles. Thus, to construct an octagon on a given side, AB, Fig. 117, prolong AB, and draw AF perpendicular to AB at A. Then, either by bisecting the right angles *aAf*, or by a 45° triangle, draw AC, and make it equal to AB. In like manner draw BD. Next find the centre, O, of the octagon, which can be done in several ways. Thus, AO bisects the angle BAC, and BO bisects ABD. *d*O is** perpendicular **to AB at its middle point, and *c*O is parallel to AB** through *c*, which is found by making **Cc perpendicular to *a*AB, and** equal to half of AB. *Any two* **of these four lines, AO, BO, *d*O,** *c*O, meet at the centre, O, of the **octagon, which is also the centre** of the circle of radius, OA, which **contains the octagon. By** drawing this circle, the remaining **sides can be found by laying** off AB five times from C around to D.

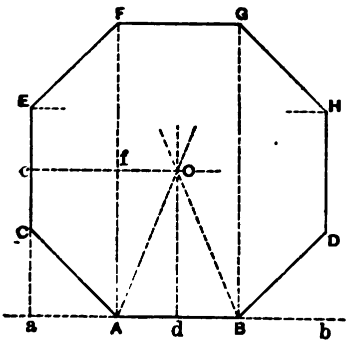

**Fig. 117.**

But the various checks upon the construction of each point make other ways convenient and accurate. Thus, *Af*, BO, and a parallel to AC through D, all meet at F, and EF is parallel to BD. The other vertices can be similarly found.

**Practice.**—1. What is the measure of the angle AO*d* in degrees?

2. What are the measures of the angles AO*c*? CA*f*? AFB? AOF?

3. Construct an octagon, having given one side = 1 inch, by finding and using the centre of the octagon as soon as one other side (as AC, Fig. 117) is found.

4. Construct an octagon, having given one side = 1 inch, without using the centre.

**235.** *The Octagon and the Square.* — The octagon has no especially simple, interesting, or useful relations to any of the preceding polygons, except the square from which it is easily derived, as follows: *fedh* being a square, and *eh* and *fd* its two diagonals intersecting at O, draw four quadrants, as shown, having the corners of the square for their centres, and the half-diagonals, as O*h*, for a radius. Then will AB = BC = CD, etc., be the sides of an octagon; for, by construction, O*h*B is an isosceles triangle with an angle of 45° at *h*. Hence *h*BO = 67½°. But by the symmetry of the figure, relative to *fd*, the angles CB*d* and BC*d* each equal 45°. Hence ABC = 135°, and OBC therefore = 67½°. Finally, OA = OB = OC, etc., all being chords of equal arcs, and OB bisects ABC. This shows that the triangle AOB = BOC (supposing AO and CO to be drawn), and that each is 45° angled at O. They are therefore equal. Therefore AB = BC, etc., and the figure ABDG is a regular octagon.

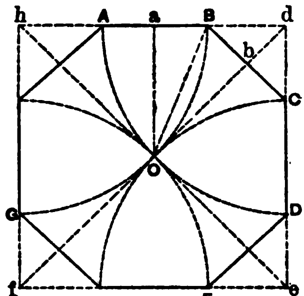

Fig. 118.

**Practice.** — 1. Having given one side, AB, of an octagon, describe a square on that side, draw the diagonals of that square, and lay off *half* of one of them from *each* end of the given side, and *on* that side produced. On the whole line, CD, thus formed, describe a square. By means of these two squares, complete the octagon.

2. Make and cut out four equal octagons; place them together about a point, as *d*, Fig. 118, as a centre (one on AB, and one against CD). What figure on BC, as one side, will *d* be the centre of? Octagons and what other regular figure will therefore combine in patch or tile work?

3. Repeat the operation, beginning by making the radii, *d*O, etc., greater than ½ *df*, and so making an octagon, regular, except that BC, DE, etc., will be smaller than AB, CD, etc.

## OTHER POLYGONS.

**236.** Such polygons as there is commonly any occasion to construct, can be made from the ones now described, either by bisecting the arcs subtended by their sides, or by the method of (227).

Thus, beginning with polygons of the fewest sides, — viz., 3, 4, and 5, or the equilateral triangle, square, and pentagon, — take first the regular inscribed polygon of three sides. By bisecting the arcs subtended by its sides, and then bisecting the half-arcs, and so on (that is, by successive bisection), we shall find polygons of 6, 12, 24, etc., sides. By repeated bisection of the arcs subtended by the sides of a square, we shall find polygons of 8, 16, 32, etc., sides. By repeated bisection of the arcs on the sides of a pentagon, we shall find polygons of 10, 20, etc., sides. Assembling these in order, we have polygons of 3, 4, 5, 6, 8, 10, 12, 16, 20, 24, 32, etc., sides.

**237.** Occasionally, also, a polygon can be simply found when we already know two others. Thus, the side of a hexagon is the chord of 60°; the side of a decagon, or polygon of ten sides, is the chord of 36°. Then, if an inscribed hexagon and decagon have one vertex common, the *chord of the difference of the arcs* subtended by their sides will be the side of a regular polygon of 15 sides (quindecagon); for $60° - 36° = 24°$, and $15 \times 24° = 360°$.

**238.** If we could only *trisect* an angle or arc geometrically, — that is, by operations with the ruler and compasses, and founded on the geometrical principles of the straight line and circle alone, — there would be a purely geometrical construction for other polygons.

Let us see a simple example of the kind of difficulties that are met with in attempting the trisection of an angle. Let *aoc*, Fig. 119, be the given angle, and suppose that *aob* were the third part of *aoc*. Having the circle of centre, *o*, and any radius, *oa*, draw *bd* parallel to *ao*, and *cd* to meet *ao* produced at *e*. Then the angles *aob* and *doe* are equal, being measured by equal arcs. Also, angle $cdb = \frac{1}{2}$ angle *boc*, because they both stand on the same arc, *bc*, while the vertex *d* is on the circumference, and the vertex *o* at the centre (190). Hence angle *bdc* = angle *aob*. But the angle *bdc* also equals the angle *deo* (129). Hence angle *deo* = angle *doe*, which makes the triangle *eod* isosceles, and *de* = *do* the radius of the circle.

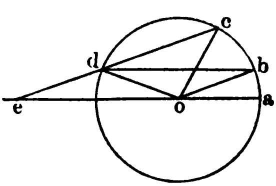

Fig. 119.

The problem of trisecting *aoc* thus requires the drawing of *cde* so that *de* shall be equal to the radius of the circle employed.

**239.** Now, although it is impossible to do this by elementary geometry, this problem affords an excellent example of a method which is often very useful, and in practice as accurate as if only the straight line and circle were used. This method is called that of trial curves.

Thus, let ABC, Fig. 120, be a given angle to be trisected. With the vertex, B, of the angle as a centre, and any convenient radius, describe an arc intersecting both sides of the angle, as at C and A, and extend it considerably, as shown at A*ad*. Then, knowing that when the angle is trisected, *re* will be equal to the radius BC, draw a number of secants (178) from A, and on them

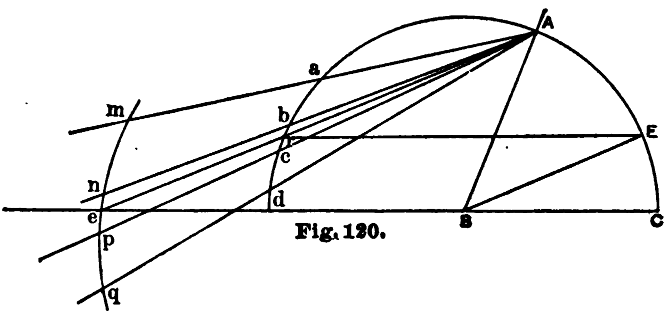


Fig. 120.

lay off *am*, *bn*, *cp*, *dq*, etc., each equal to BC. The curve sketched carefully through the points *m*, *n*, *p*, *q*, thus constructed, will evidently contain that point, *e*, of CB produced, at which, if *e*A be drawn, *er* will be equal to BC. Then, drawing *r*E and EB, the angle EBC will be one-third of ABC.

The trial curve *mnq* is called the *locus* of all points at a distance from the circle equal to the radius, and on lines from A.

A few angles can be trisected geometrically. Thus, the angle of 60° (188) leaves from 90° *one-third* of 90°, or 30°. Also *bisect* an angle of 90°, giving 45°; and 45° from 60° leaves 15°, or *one-third* of 45°.

**Practice.**—1. Trisect a right angle.
2. Trisect an angle of 45°.

**240.** Every regular polygon consists of as many isosceles triangles as it has sides, and *each* of these triangles gives *two right angles* as the sum of its angles. But the sum of all the angles at the centre of the polygon is four right angles. Hence the sum of the angles of the polygon is as many times two right angles as the figure has sides, minus four right angles. Thus, the sum of the angles of a hexagon = 6 × 180° − 360° = 1080° − 360° = 720°.

Any *one* angle of a regular polygon will be equal to the *sum* of the angles divided by the *number* of angles. Thus, the angle of a decagon is (10 × 180° − 360°) ÷ 10 = 144°.

**Practice.** — 1. How many degrees in the angle of a dodecagon (12 equal sides)?

2. Place three equal dodecagons, each two with one side common, and show by their angles that an equilateral triangle will fill the space left between them.

**241.** Irregular polygons are of infinite variety. Country fields of all shapes, sizes, and number of sides are the most familiar practical examples.

An irregular polygon, Fig. 121, cannot generally be divided into equal, much less into isosceles, triangles. They will generally be *unequal scalene* triangles.

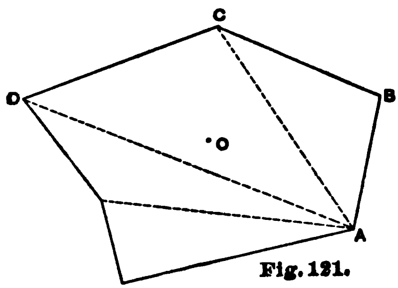

Fig. 121.

Every polygon can be divided, by diagonals, AC, AO, etc., from any *one* of its angles, into *two less triangles than it has sides.* Thus, Fig. 121, which has *six sides*, can thus be divided into *four triangles.* The sum of the angles of all these triangles is evidently equal to the sum of the angles of the polygon. Hence this sum is 2 right angles taken 2 *less* times than the figure has sides.

We could have reasoned in the same way upon the regular polygons; and we could also have reached the same result as just found by taking *any* point, O, within Fig. 121, and making it the common vertex of as many triangles as the figure has sides, and by then reasoning as in (240).

**Practice.** — 1. Prove, by the method of Fig. 121, that the sum of the angles of a regular pentagon is 540°.

2. Prove, by triangles having O for a common vertex, that the sum of the angles of Fig. 121 is 720°.*

* The word GEOMETRY means the measuring of the ground; but it is now extended to mean the knowledge of all space relations. Such knowledge, then, has applications to all

**242.** Since mechanical invention requires one to be familiar with many different shapes, and how to put them together to make what is desired in machines, buildings, or other objects, practice in geometrical tile-work will be found very useful as well as interesting. Here are a few examples.

**Practice.** — 1. Having tiles 3 inches square, with halves of the same, plan a hearth of 3 rows of 11 tiles in a row, and set diagonally, and find the length and width of the whole in the nearest number of whole inches.

2. Having, also, pieces 6 inches by 1 inch, and 6 inches by ½ inch, and 3 inches by 6 inches, plan a border 4½ inches wide to the hearth, and find the length and width of the whole.

3. Using the same pieces, excepting the half-squares in the central portion, plan the hearth when the squares are set with their edges parallel to the sides of the hearth.

4. Wishing to set squares diagonally in a band 3 inches wide, what will be the length of the side of the square whose diagonal is 3 inches?

5. Having octagonal tiles 3 inches across, with the alternate sides 1½ inches long, what will be the length (very nearly) of the other sides?

6. Fill a space 24 inches by 48 inches with the tiles just described, and squares, and half-squares with sides 1½ inches long.

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practical questions of size, form, and position, in every branch of industry and department of life where, also, *other* knowledge is required, — to astronomy, navigation, gunnery, architecture, mechanics, surveying, etc. Hence, *first*, it is impossible to introduce here examples of *all* these applications; and therefore, *second*, it seems best to introduce *none*, further than by an occasional passing remark on applications so purely geometrical as regular tile-work (221, 222). The division of land is a branch of surveying requiring a knowledge of surveying instruments and operations, though it consists essentially of the division of irregular polygons in various ways and under various conditions. But, though founded on geometry, it is not a part of it; and, further, surveying is no part of farming. Hence learners who may be interested in land, are commended to works on surveying for information about the division of land; and the like may be said of other branches.

# CHAPTER VI.

## AREAS.

### Areas of Rectilinear Figures.

**243.** The difference between surface and area is simply this, that area expresses the *measure* of the surface in some unit of area generally agreed upon, as square inches, square feet, square miles; or square meters, etc.

In inquiring about the area of a surface, we ask, How *long* and how *wide* is it? but length and width, being the two *dimensions* (25) of a surface, are at right angles to each other, as are also the sides of a square. Hence the *square, each of whose sides is some unit of length*, is at the foundation of our ideas of area, and hence the study of area naturally begins with that of square-cornered figures, or squares and rectangles.

**244.** For example, let the distance 01, Fig. 122, represent one inch. Then 011$a$ represents 1 square inch; and as the rectangle 01$b$8 contains 8 squares, each equal to 011$a$, its area is 8 square inches.

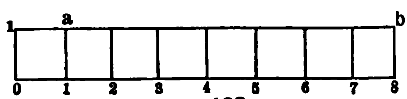


Fig. 122.

**245.** Again, if, in Fig. 123, we suppose 01 to be 1 inch, the rectangle ABCD is 5 inches long and 3 inches wide, and hence evidently contains 3 *rows* of square inches of 5 square inches in each row, making 15 *square inches* in the rectangle. But 15 is the product of the numbers representing the inches of length and width. Hence the familiar

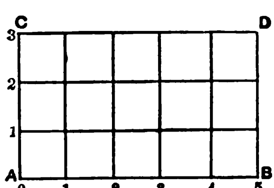


Fig. 123.

RULE. — *To find the area of a rectangle, multiply its length by its width, and the product will be its area.*

In this rule it is always understood that the two dimensions are to be expressed in units of length of the *same denomination.* The area will then be expressed in *square* units of the same denomination.

If the dimensions are not *given* in the same denomination, they must be exchanged for equal ones that are. Thus, if a rectangle be 2 *feet* long and 16 *inches* wide, we must call it 24 inches long and 16 inches wide, or 2 feet long and $1\frac{1}{3}$ feet wide.

**246.** By keeping in mind the foundation-thought of so many rows of square units, at so many in a row, as in Fig. 123, we shall not be at all perplexed by fractions in either or both of the given dimensions. Thus, if AB represents a rectangle $3\frac{1}{2}$ feet long and $2\frac{1}{4}$ feet wide, it evidently contains $2\frac{1}{4}$ *rows* of square feet, with $3\frac{1}{2}$ square feet in each row. The total number of square feet is therefore $2\frac{1}{4} \times 3\frac{1}{2}$, or $\frac{9}{4} \times \frac{7}{2} = \frac{63}{8} = 7\frac{7}{8}$ square feet. Or, if we prefer decimal fractions to common fractions, $2\frac{1}{4} = 2.25$ and $3\frac{1}{2} = 3.5$, and the area $= 2.25 \times 3.5 = 7.875$ square feet, which is the same as $7\frac{7}{8}$ square feet.

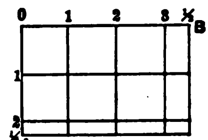


Fig. 124.

**247.** The last example, with its decimal divisions of the unit, shows us all that is peculiar to the calculation of areas expressed in the metric system. Thus, if a floor be 3.06 meters long (that is, 3 meters and 6 centimeters) and 2.84 meters wide, its area will be found as in any other case of multiplication by decimals. Thus, $3.06 \times 2.84 = 8.6904$ square meters, or 8 square meters 69 square decimeters and 4 square centimeters, or 8 square meters and 6,904 square centimeters, whichever of these three ways we please to express it; for a decimeter is $\frac{1}{10}$ of 1 meter, and a square decimeter is $\frac{1}{10} \times \frac{1}{10} = \frac{1}{100} = .01$ of a square meter. Likewise, as a centimeter is $\frac{1}{100}$ of 1 meter, a square centimeter is $\frac{1}{100} \times \frac{1}{100} = \frac{1}{10000} = .0001$ of a square meter.

**Practice.** — 1. What is the area of a square each of whose sides is 3 feet long? 9, what? 1, what?

2. What is the area of a square each of whose sides is 4 feet 6 inches long?

3. What is the area of a square each of whose sides is 35 centimeters?

4. What is the area of a board 30 inches long and 20 inches wide?

5. What is the area of a closet floor 48 inches long and 28 inches wide?

6. What is the area of a doorway 2 feet 10 inches wide and 7 feet 6 inches high?

7. What is the area of a window 82 centimeters wide and 1 meter 76 centimeters high?

8. What is the area of a lot 124 feet front and 236 feet deep?

9. What is the area of a town 3 miles 32 rods long and 2½ miles wide?

**248.** If a triangular portion, ACc, Fig. 125, be cut off from one end of a rectangle, ABcd, and transferred to the other end, as at BDd, it is evident that no change has been made in the area of the figure; that is, in the amount of surface contained in it. There still continue to be 4 rows of squares of 7 squares in a row; for the two parts, A1a and Bbc, just make, with the 6 undivided squares, 7 squares, and the like is true for the other rows.

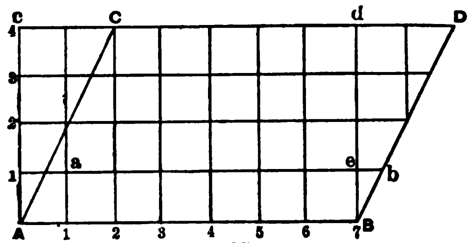


Fig. 125.

**249.** The new figure, ABCD, Fig. 125, is called a *parallelogram*, and is defined by saying that its *opposite* sides are *equal and parallel*.

Accordingly, from what has just been shown, the *area of a parallelogram* is found by multiplying the *length* of one of its sides (which expresses the number of square units in one row, Fig. 125) by the perpendicular distance between that and the opposite side (which expresses the number of rows).

The side taken is commonly called the *base*, and the perpendicular mentioned is called the *altitude*. Hence the rule is more briefly expressed thus: *The area of a parallelogram is the product of its base by its altitude.*

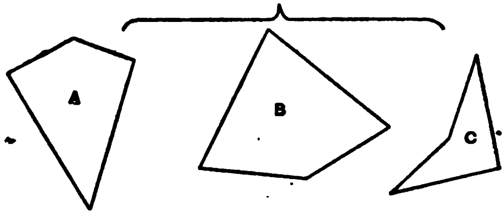
**Practice.**—1. Find the area of a parallelogram whose base is 15 feet, and altitude 9½ feet.

2. Find the area of a parallelogram of a base 6 meters 16 centimeters, and an altitude of 4.36 meters.

**250.** Speaking more exactly, the figure last mentioned is an oblique-angled parallelogram. The rhombus, square, and rectangle are also all parallelograms; the two former *equal-sided*, and the two latter *right-angled*.

**251.** This beginning of the study of areas is limited to parallelograms, — that is, to figures having two pairs of parallel sides, — because only such have a *uniform* length and height.

Fig. 126.




Other four-sided figures, together with those already mentioned, are known by the general name of *quadrilaterals*, meaning four-*sided* figures; or *quadrangles*, meaning four-*angled* figures.

The following table exhibits them all to the mind systematically: —

| | | | |
|---|---|---|---|
| All QUADRILATERALS or QUADRANGLES are either | *Parallelograms* (all the opposite sides parallel) | Right-angled | Squares, Fig. 95. |
| | | | Rectangles, Fig. 102. |
| | | Oblique-angled | Rhombus, Figs. 103, 104. |
| | | | Rhomboid, Fig. 125. |
| | *Trapezoids* (one pair of opposite parallel sides), Fig. 131. | | |
| | *Trapeziums* (no two sides parallel), Fig. 126. | | |

**252.** Triangles of every variety of form, whether

| | | |
|---|---|---|
| equilateral,<br>isosceles,<br>scalene, | or | right-angled,<br>obtuse-angled,<br>acute-angled, |

can be formed of the halves of parallelograms. We shall therefore next show how to find the areas of triangles. Then, as all other figures are composed of triangles, we shall show how to find their areas by finding the areas of the triangles of which they are composed.

60°

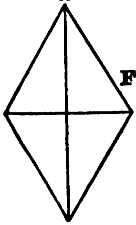


Fig. 127.

Thus, the rhombus, Fig. 127, having angles of 60°, is divided by its short diagonal into *equilateral* triangles; and any other rhombus is divided by both diagonals into *isosceles* triangles, *obtuse-angled* by the longer diagonal, and *acute-angled* by the shorter one. A square is divided by its diagonals into isosceles right-angled triangles, Fig. 128; while rectangles, Fig. 129, are divided in the same manner into *scalene* right triangles. Finally, the longer diagonals of a parallelogram, Fig. 130, divides it into *obtuse-angled scalene* triangles, and the shorter one into sometimes *acute-angled*, and sometimes *obtuse-angled*, scalene triangles. Moreover, it is evident from the figures that each figure is divided by each one of its diagonals into two equal triangles, each of which, therefore, has half the area of the whole figure.

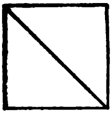


Fig. 128.

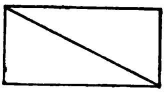


Fig. 129.

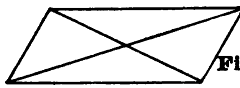


Fig. 130.

**253.** We therefore have the following principles:—

1°. Every parallelogram is divided by each of its diagonals into two equal triangles, each of which is therefore half of the original figure.

2°. Every possible form of triangle can be found by thus dividing different parallelograms.

3°. Therefore, as the area of every parallelogram is the product of its base by its altitude, the area of a triangle, every triangle being half of some parallelogram, is the *product of its base by half its altitude;* or, which is the same thing, half the product of the base and altitude.

See, for illustration, Fig. 125, where the rectangle A2Cc contains 8 squares, and the triangle A2C, which is half of it, evidently contains squares and parts of squares, which when put together would make 4 squares.

*Any* side of a triangle can be taken as its base, and then the perpendicular to that side, from the opposite angle, will be the corresponding *altitude* of the triangle (146).

**Practice.**—1. The base of a triangle being 30 feet, and its altitude 16 feet, what is its area?

2. The sides of a rectangle being 150 feet and 130 feet, what is the area of the triangle which is its half?

3. Triangles having equal areas are therefore equivalent (149); then, if the base and altitude of one triangle are 24 feet and 9 feet, and the base of an equivalent triangle is 28 feet, what will be its altitude?

4. What is found by dividing the area of any triangle by its base? by its altitude?

5. A triangle contains 300 square feet, and its altitude is 15 feet. What is its base?

6. A rose-garden in the form of a rhombus has a longer diagonal of 120 feet, and a shorter one of 72 feet. Then, 1st, into *how many* and *what kind* of triangles do its diagonals divide it? 2d, What is the base and altitude of each? 3d, What is the area of *each*? also of *all*; that is, of the rhombus? 4th, Compare this area with the product of the given diagonals, and thence make a rule for finding the area of a rhombus when its two diagonals are given.

7. Find the length of the circumference of the garden in the last example.

**254.** Having now learned how to find the areas of triangles, *the area of any irregular quadrilateral* can be found by dividing it into two triangles, and by adding together the areas of both triangles. To begin with the trapezoids,—

The area of a triangle equals $\frac{\text{the base} \times \text{the altitude}}{2}$, which equals *the base* $\times$ $\frac{\text{the altitude}}{2}$, or *the altitude* $\times$ $\frac{\text{the base}}{2}$.

Let us consider the last expression; viz., the altitude multiplied by half the base: then in the trapezoid, *abcd*, Fig. 131, the area of the triangle *abc* equals the altitude *ce* multiplied by half the base *ab*, and the area of the triangle *dbc* equals the altitude *bg* multiplied by half the base *cd*. Adding these, the area of the trapezoid equals the altitude *bg* multiplied by half the sum of *ab* and *cd*; that is, the rule for finding the area of a trapezoid is this:—

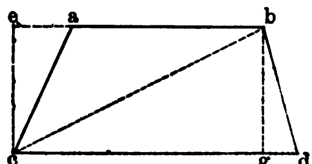


Fig. 131.

RULE.—*Multiply the half sum of its parallel bases by the perpendicular distance between those bases.*

Thus, suppose *ab* were 10 feet, *cd* were 14 feet, and *bg* or *ce* were 6 feet; then the area of the trapezoid would be $(5 + 7) \times 6 = 12 \times 6 = 72$ square feet.

**Practice.**—1. The length of the bases of a certain trapezoid are 7 yards and 5 yards, and its altitude is 2 yards 2½ feet. What is its area?

2. Draw any trapezoid, take the necessary measures as accurately as possible, and find its area.

3. A trapezoidal city lot has parallel sides of 40 and 70 feet, and a perpendicular between them of 18 feet. What is it worth at $14.65 per square foot?

4. A man remembers that the trapezoidal floor of his office contains 324 square feet. He measures the parallel widths, and finds them 18 feet and 14 feet. What is the length of his office?

5. A board fence 15 meters long is 2 meters high at one end and 1.36 meters high at the other end. How many square feet of boards are there in it?

6. A boy remembers that his garden is 42 feet long and 24 feet wide at one end, and contains 1,260 square feet. How wide is it at the other end?

**255.** In the same way the area of any irregular quadrilateral, as the trapezium, *abcd*, Fig. 132, is found; viz., by dividing it into two triangles, as *abc* and *adc*, and adding the areas of those triangles. In this way land can be measured.

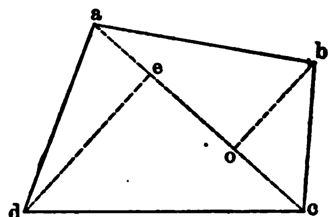



Fig. 132.

**Practice.**—1. Suppose $ac = 92$ feet, $bo = 28$ feet, and $de = 34$ feet; what is the area of the trapezium?

2. Drive four stakes so as to form a trapezium. Measure one of its diagonals, and the altitudes of the triangles (Fig. 132) into which it divides the trapezium, and find the area of the trapezium.

3. Suppose that a clump of bushes prevented measuring either diagonal in the last example. Measure any two opposite sides, and the perpendiculars to them from diagonally opposite corners (from *a* to *cd*, and from *c* to *ab*, Fig. 132), and find the area.

**256.** If the area of *one* triangle $= \frac{1}{2} B \times A$, where B is the base and A the altitude, the area of *seven* equal triangles will be

$$\tfrac{1}{2}7B \times A = 7B \times \tfrac{1}{2}A.$$

And the like is true for any other number of equal triangles.

But any regular polygon is composed of as many equal triangles as it has sides, and their equal altitudes are equal to the radius of the inscribed circle (197, 204). Thus (Fig. 133), the regular hexagon ABDF is composed of *six* triangles, all equal to ABO, and all having altitudes equal to OG. Hence the area of the hexagon = 6AB × ½OG, or the perimeter, ABCDEF, multiplied by half the radius of the inscribed circle. How much this radius is, cannot now be found in ways which the learner can understand, except in a few cases. In the other cases it must either be measured as exactly as possible with a finely divided rule, or taken on trust from the following table, in which the sides of the polygon are each equal to 1.

Then the radius of the inscribed circle is for the

| | | | |
|---|---|---|---|
| Equilateral triangle | 0.2887 | Regular octagon | 1.2071 |
| Square | 0.5 | Regular enneagon (9 sides), | 1.3937 |
| Regular pentagon | 0.6882 | Regular decagon | 1.5388 |
| Regular hexagon | 0.866 | Regular dodecagon | 1.866 |
| Regular heptagon | 1.0383 | | |

**257.** In the case of the hexagon, since it is composed (Fig. 133) of six equal equilateral triangles, if the side $ab = ac = bc$ (Fig. 134) is 1, then $ad = \frac{1}{2}$, and we have $(ac)^2 = 1$, $(ad)^2 = \frac{1}{4}$, and $(cd)^2 = \frac{3}{4}$, whence $cd = \frac{1}{2}\sqrt{3} = 0.866$.

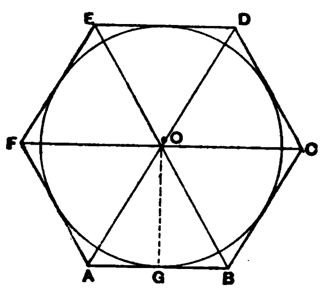


Fig. 133.

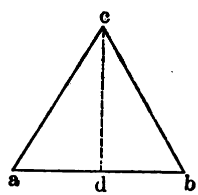


Fig. 134.

The radius of the inscribed circle of a square is obviously equal to half the side of the square. The pupil may find that of the inscribed circle of an equilateral triangle if he pleases, though it is unnecessary, as a triangle usually has its base and altitude given when its area is to be found.

When the side of the polygon has any other value than 1, the value of the radius of the inscribed circle will be found by multiplying the radius given in the table, by the given value. Thus, if the side of a hexagon is 7, the radius of its inscribed circle, or common altitude of its equal component triangles, will be 7 × 0.866.

**Practice.**— 1. Find the area of a regular pentagonal lot each of whose sides is 56 feet.

2. Suppose a walk 6 feet wide bounded by the perimeter of the pentagon in the last example, what will be the *side* and the *area* of the pentagon within the walk, and what the difference of the areas of the pentagons, or the area of the path? The common altitude (A) of the outer pentagon ÷ A − 6 feet, the common altitude of the inner one = the side (56 feet) of the outer ÷ the side of the inner. Knowing this equality of ratios (which will be obvious to sight on drawing the figure), the required side of the inner one can be found, and thence its area.*

3. Find the area of an octagon each of whose sides is 12 feet.

4. How many square inches in a regular octagon whose perimeter is 24 inches?

5. The side of a regular octagon is 6 feet. What is the side and area of the square from which the octagon was formed by cutting off the corners? See (211).

6. The chancel of a church is a half-decagon, each of whose sides is 9 feet. What is its area?

7. How many whole hexagonal tiles, the sides of each being 6 inches, can be placed as in a floor 9 feet 3 inches wide by 32 feet long?

8. One side of a regular octagon being 3 feet, what is the side of a square having the same *area* as the octagon?

9. Find the area of a square by multiplying the length of its side, 12 feet, by itself; also by multiplying its perimeter by half the radius of its inscribed circle.

10. Find the area of the rectangle which will just contain a hexagon each of whose sides is 2 feet.

11. Draw a large pentagon by inscribing it in a circle, measure its side and the radius of its inscribed circle, and find its area.

12. Do the same with a regular octagon.

### Area of the Circle.

**258.** The more sides a polygon has, the shorter they must be if it is inscribed in a given circle, as may be seen by any of the previous figures of inscribed polygons.

If, then, the number of sides be indefinitely great, they must be indefinitely short. Thus, the perimeter of a polygon finally becomes the circumference of a circle, considered as a polygon of an *infinite*, or *unthinkably great*, or *indefinitely great*, number of sides.

Hence it is easy to see that the rule for finding the area of a polygon holds true for the circle, considered as composed of an

* The pupil is here supposed to have acquired some elementary notions of ratio and proportion from his arithmetic.

infinite number of isosceles triangles whose bases are infinitely short, and whose common altitude is the radius of the circle. The sum of all the bases forms the circumference of the circle; hence, to find the area of a circle, multiply its circumference by half its radius.

**259.** But now arises the question, How shall we find the circumference when the radius or the diameter of a circle is given, as it commonly is?

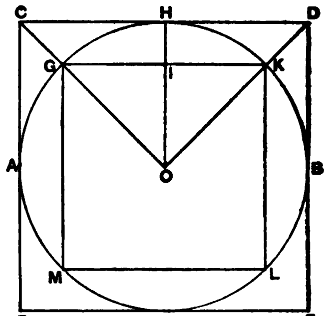


Fig. 135.

When there is no number that will divide (that is, measure) two lines, they are said to be incommensurable. The diameter (or radius) and the circumference of a circle are two such lines. But we know, by any diagram of the the three figures, that the circumference of a circle is *less* than that of its *circumscribed* polygon, and *more* than that of its *inscribed* one.

**260.** To illustrate, let the diameter, AB, of a circle = 1 foot. Then the perimeter, CDEF, of the circumscribed square = 4 feet.

$$OG = \tfrac{1}{2} = OH = HC;$$

and, by (211),

$$OC = \sqrt{OH^2 + HC^2},$$

$$= \sqrt{\tfrac{1}{4} + \tfrac{1}{4}},$$

$$= \sqrt{\tfrac{1}{2}} = \tfrac{1}{2}\sqrt{2}.$$

But OCH and OGI are evidently similar triangles, so that (153) GI is the same part of CH that OG is of OC, or

$$\frac{OC}{OG} = \frac{CH}{GI}, \text{ or } \frac{\tfrac{1}{2}\sqrt{2}}{\tfrac{1}{2}} = \frac{\tfrac{1}{2}}{GI}, \text{ or } \sqrt{2} = \frac{\tfrac{1}{2}}{GI}, \text{ or } GI \times \sqrt{2} = \tfrac{1}{2},$$

or, finally,

$$GI = \frac{\tfrac{1}{2}}{\sqrt{2}} = \frac{1}{2\sqrt{2}} = \tfrac{1}{4}\sqrt{2}.$$

Then the perimeter, GKLM, of the inscribed square $= 8 \times$ GI $= 2\sqrt{2} = 2.82842$. Hence the length of the circumference whose radius is OA lies between 4 feet and 2.82842 feet.

**261.** Next, we are able to find the length of the perimeter of inscribed and circumscribed polygons of twice as many sides. Thus, —

Let GK, Fig. 136, be the side of the inscribed square, and CD that of the circumscribed square of the circle of radius OH, as in the last figure. GH is evidently the side of the inscribed regular octagon, and $GH^2 = GI^2 + HI^2$. But HI = OH − OI, and in the right triangle OIG, OI = GI. Hence the quantities GI and HI, from which GH is found, are both known, and GH can therefore be easily found.

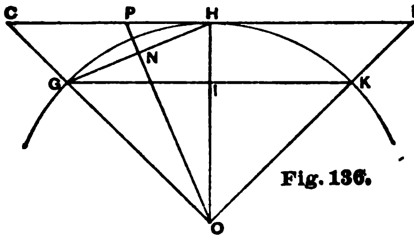


Fig. 136.

Again, as ON is drawn perpendicular to GH, it bisects it and the angle, GOH, of 45°. Hence twice POH would be 45°, or the proper angle at the centre of an octagon; and PH is half the side of the regular circumscribed octagon.

Now, $PH^2 = HN^2 + PN^2$. Here we know HN, which is half of GH, found above. PN is found by recalling (153) that PN is the same part of HN that HN is of ON, so that $\frac{OH}{ON} = \frac{HN}{PN}$. But $ON^2 = OH^2 - HN^2$, by which we can find ON. Now, remembering that $OH = \frac{1}{2}$, $OI = GI = \frac{1}{4}\sqrt{2}$, $HI = OH - OI = \frac{1}{2} - \frac{1}{4}\sqrt{2}$, we successively find GH, HN (which is half of GH), and ON. This gives us all the quantities used in finding PN; and when we know that, we can find PH and 2PH, the side of the circumscribed octagon. When the calculation, which is a little tedious though not difficult, is done, we shall find the perimeter of the circumscribed octagon to be 3.3137 feet, and that of the inscribed octagon to be 3.06146 feet.

**262.** By continuing these operations to polygons of a great number of sides, the circumference of the circle can be found exactly to as many decimal places as the perimeters of its inscribed and circumscribed polygons agree. Thus, when we come to circumscribed and inscribed polygons of 2,048 sides, their perimeters, and hence the circumference of the circle lying between them, agree to five decimal places, 3.14159, commonly written 3.1416, and sufficiently accurate for all common purposes, since the error would be only about 1 inch in the circumference of a circle a mile in diameter.

$\frac{1}{7}$ = .1428; hence we see, that, for all operations not requiring great nicety, $3\frac{1}{7}$ is a sufficiently accurate expression for the value of the circumference of a circle whose diameter = 1.

**Practice.**—1. If a wheel be 4 feet in diameter, what is its circumference?

2. How far will it go in making 12 revolutions?

3. If the wheel of a bicycle be 54 inches in diameter, how many revolutions will it make in going 1 mile?

4. Having a cylindrical vessel with hemispherical ends, I measured its circumference with a tape, and found it 10 feet 8 inches. What is the diameter of the cylinder?

5. A man walked a measured 1,000 feet five times, at an average of 3 minutes to each time. Walking as nearly as possible at the same rate, three men averaged 1 hour and 28 minutes in walking round a circular lake. What was the circumference and the diameter of the lake?

6. At 10 tacks to a foot, how many tacks will be required around the circumference of a piano stool $15\frac{1}{4}$ inches in diameter?

7. A mill-wheel 3 feet in diameter actuates another of 15 inches diameter by means of a band. How many revolutions does the latter make while the former makes 10?

8. A fly-wheel has 56 teeth, 4.8 inches apart from centre to centre of two successive teeth. What is the radius of the wheel?

9. If the moon is 240,000 miles from the earth, and its orbit circular, what is the circumference of its orbit?

10. If a man makes 27 paces of 3 feet each in going around the edge of a circular pool, what is the diameter of the pool?

11. A carriage-wheel 50 inches in diameter had an attachment which recorded 2,400 revolutions between two points. How far apart were those points?

12. If a semicircular recess is to seat 40 persons against its wall, allowing 22 inches on the wall to each person, what will be the diameter of the recess?

13. If the long arm of a weather-vane be 6 feet long, and the short arm 20 inches, what is the difference of the distances which the two ends travel in one complete revolution?

14. Calling the diameter of the earth 7,950 miles, what is its circumference?

15. If the earth makes one revolution in 24 hours, how far does a person at the equator move in one hour?

16. Three wheels have diameters of 2, 4, and 5 feet. What are their circumferences, and how do the ratios of these circumferences compare with those of their diameters?

17. A conical tower is 12 feet in diameter at the base, and 8 feet across at the top. What is its circumference at both places, and at $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{3}{4}$ of its height?

NOTE. — Work all these examples over again, substituting meters and parts thereof for feet and inches.

**263. EXAMPLES ON AREAS.** — For convenience of expression, the number 3.14159 . . ., by which the diameter of a circle is multiplied in order to find its circumference, is called $\pi$ (pi). Then, if the diameter of a circle be called $d$, its circumference $= \pi d$. But $d = 2r$, calling $r$ the radius, and the circumference $= \pi d = 2\pi r$.

Now, the area = the circumference multiplied by half the radius (258), or $A = C \times \frac{r}{2} = 2\pi r \times \frac{r}{2} = \pi r^2$. Thus we see that we can find the area of a circle if we only know the radius. Or, if we choose to use the diameter, $\frac{1}{2}$ the radius $= \frac{1}{4}$ the diameter, and the area $= \pi r^2 = \pi d \times \frac{d}{4} = \frac{1}{4}\pi d^2$. Here note, that, as $r$ is the only changeable or (as it is commonly called) variable quantity in the expression $\pi r^2$ for the area of a circle, the area varies as the *square* of the radius. This means, that if, for example, the *radius* of one circle be 10 times as great as that of another circle, the *area* of the first will be, not 10, but 100 times that of the second. And the like is true of all areas.

**Practice.** — 1. Find the area of a circle whose radius is 3 feet.

2. Find the area of a circle whose diameter is 5 feet.

3. Find the area of a circle whose circumference is 27 feet.

4. Find the radius of a circle whose area is 36 square feet.

5. Find the diameter of a circle whose area is 20 square feet.

6. Find the circumference of a circle whose area is 56 square feet.

These six examples embrace the changes possible in the formula $A = C \times \frac{r}{2}$. Practical questions consist of varied applications of them, combined with conditions as to cost, equal distances, comparison of areas, etc., and can be made by the pupil for himself, with the following as a guide: —

7. A circular tower 25 feet in diameter stands in a circular grass plot 116 feet in diameter. Find the area covered by the tower, and that of the grass.

8. In the centre of a square is a circle of tile-work 20 feet in diameter, and costing $1.42 per square foot. What is the cost of the circle?

9. The radii of two circles are 2 feet and 4 feet. Find and compare their *areas.*

NOTE. — Similar areas, as circles, etc., are to each other as the squares of corresponding lines in each.

10. The areas of two circles are 49 square feet and 81 square feet. Find and compare their radii (or diameters, or circumferences).

NOTE. — Corresponding *lines,* in circles or other similar figures, are to each other as the square roots of the areas.

11. What is the diameter of a circle which contains one acre?

NOTE. — Change the acre into feet, yards, or rods, which names are used both in long and square measure (245).

12. A circular park is to be bordered by a drive 3 miles long, measured in the centre of the roadway, which is to be 64 feet wide. How many acres will there be in the park, and what area will the roadway cover?

13. Out of a uniform sheet of brass a square and a round plate of equal areas of 225 square inches are punched. Find and compare their perimeters.

14. Out of the same sheet of brass an equilateral triangle, an octagon, and a circle of equal perimeters of 24 inches are punched. Find and compare their areas.

**264.** The area of any sector, as *abc*, Fig. 137, equals the length of its arc, *ab*, multiplied by half its radius, *bc*. For example, let the diameter, $2bc$, = 16 feet, and let *ab* be an arc of 55°. What is the area of the sector?

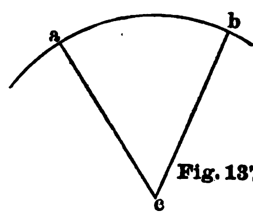


Fig. 137.

1st, The length of an arc of 1° is $\frac{\pi}{360} = \frac{3.14159}{360} = 0.0087266$, when the diameter is 1.

2d, Then the length of an arc of 1°, when the diameter is 16 feet, is 16 times as great, or 0.1396256.

3d, Then the length of 55°, the diameter being 16 feet, is $0.1396256 \times 55 = 7.679408$ feet.

4th, Finally, the arc *ab* being 7.679408 feet, and the diameter being 16 feet, the area of the sector is $7.679408 \times \frac{8}{2} = 30.717632$.

15. What is the area of a sector of 62° when the radius is 15 feet?

16. Two radial paths of a circular park are 54° apart. What area is included between their centre lines if the park be 500 feet in diameter?

**265.** Otherwise, remembering that a *sector* of a circle is the part included by an arc and its two extreme radii, its area is

**the same part of the whole circle to which it belongs, that the number of degrees in its arc is to 360°.**

17. What is the area of a sector of 45° in a circle of 10 feet radius?

NOTE. — Remembering that a *segment* of a circle is the area between an arc and its chord, its area is the difference between the sector having the same chord, and the isosceles triangle whose base is that chord, and whose sides equal the radius of the sector.

18. In a circle 3 feet in diameter, what is the area of each of the equal segments left by an inscribed equilateral triangle?

19. By a square?

20. By a regular hexagon?

21. By a regular octagon?

22. If you were to buy a circular lot 108 feet in diameter, at 24 cents a square foot, which would be cheaper, to take $\pi = 3\frac{1}{7}$, or $\pi = 3.1416$, in computing the area? and how much?

23. How many times as much water will flow from a reservoir through a 12-inch pipe as through a 4-inch pipe?

24. How much land will a locomotive roundhouse cover, allowing 50 feet length for each engine and its tender, outside of a central turn-table 40 feet in diameter, and 2 feet thickness of the outer walls?

NOTE. — Substitute metric long or square measure at pleasure in any or all of the foregoing examples.

# CHAPTER VII.

## LINES AND PLANES IN SPACE.

### Positions.

**266.** Various solid bodies have already been briefly described in the first chapter of this book: and in all the study of plane figures in the last three chapters, we have, after all, been studying solids only in an indirect manner, for solids are bounded by lines and surfaces, and many of these surfaces are plane figures; so that what now really remains to do is only to observe a few more things concerning the *forms*, *positions*, and *measures* (10) of solids, that could not be so well understood before we had studied the lines, angles, and figures that go to make up the definite forms of solid bodies.

**267.** In studying lines and plane figures separately, we have considered them as lying in some one plane; but solid bodies are bounded by lines and figures having various positions not all in one plane, and we will therefore begin again the study of magnitudes in space by noticing some useful facts about *lines and planes in space*.

As a plane is of unlimited extent (42), it can only be indicated to the mind in a diagram by some plane figure, and better if one or more lines are represented as in the plane.

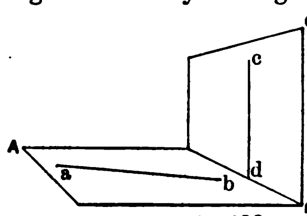


Fig. 138.

Thus, Fig. 138 may represent a horizontal plane, AB, as a floor, with a line, *ab*, in it, and CB a vertical plane with a line, *cd*, in it.

**268.** Any number of planes can all contain some one line; for the face of a door, for example, or of the lid of a box, is a plane, and every position of the door or lid makes a new plane, all containing the one line connecting the hinges on which the door or lid turns. Thus, also, the planes LM, N, PQ, and RS, Fig. 139, all contain the line AB, which is, indeed, their common intersection.

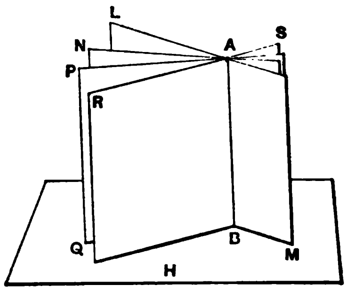


Fig. 139.

If all these planes are vertical, as are the leaves of a book set up on end and partly open, their intersection, AB, will be vertical also, or perpendicular to a horizontal plane, H. If, now, the whole be of tin plates fastened together into one whole, the line AB, and all the planes, LAB, NAB, etc., which contain it, will be perpendicular to H, in whatever way the whole body is turned; that is, if a line is perpendicular to *any* plane, all planes containing that line will be perpendicular to that plane.

**269.** Planes are said to be parallel, just as lines are, when they are everywhere at the same perpendicular distance apart. Thus, a level ceiling is parallel to a level floor; the shelves of a closet are parallel planes, and so are opposite sides of a square box.

When the planes are parallel, every line in either of them is parallel to the other one. Thus, every line in a ceiling is parallel to the floor. The parallel line is everywhere at the same perpendicular distance from the plane.

**270.** Suspend a plumb-line in a room so that it shall just touch the floor at a point *a*. Through *a* draw any number of chalk-lines, or lay any number of straight wires. All the chalk-lines will be *horizontal.* The plumb-line is *vertical*, and therefore perpendicular to all the chalk-lines; that is, *when a line is perpendicular to a plane, it is perpendicular to all lines through its foot in that plane.*

Also, at any one point in a plane there can be but one perpendicular to the plane; for if the upper end of the plumb-line were moved, while its lower end, *a*, were held fast, it would immediately cease to be perpendicular to all but one of the lines in the floor through *a*, and hence would no longer be perpendicular to the plane.

**271.** The shelves of a bookcase are parallel planes, and they meet the back and the sides of the case in parallel lines; the faces of the slats of blinds are parallel planes, and they meet the upright pieces of the frame of the blinds in parallel lines. These and many other examples show that *parallel planes meet or cut any other plane in parallel lines.*

**272.** Strings from the top of an upright pole to points on a level circular hoop whose centre is at the foot of the pole, are equal, Fig. 140; for the pole, the string, and the radius of the hoop form a right-angled triangle, ABC, of fixed size for every position of the string, since the pole, the right angle A, and the radius of the hoop remain unaltered. Hence all the positions of the string BC are equal; that is, —

**Fig. 140.**

All lines from a fixed point, and which make equal angles, as BCA, with a plane, or which meet it at equal distances from the foot of the perpendicular (AB) from the point to the plane, are equal. Also, if the angle BCA or the distance AC be made greater, BC will become longer; but if either be made less, BC will become shorter.

**273.** If *ab* and *cd* be two lines in the top of a level table, T, Fig. 141, and if *pq* be a knitting-needle held above the table, but parallel to *cd*, then *ab* and *cd* are in the same plane, T, and *pq* and *cd* are in the same plane, *pmnc*; but *no one plane* contains, or can be made to contain, both *pq* and *ab*. They are not in the same plane. But why? Notice that *ab* and *cd intersect* at *n*, and that *pq* and *cd* are *parallel*. But *pq* and *ab* neither intersect nor are parallel: that is, if two lines *either* intersect or are parallel, they *are in the same plane;* but if they *neither* intersect nor are parallel, they *are not* in the same plane.

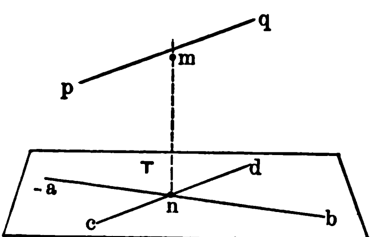


Fig. 141.

**Practice.**—1. What is a plane, and how may it be represented?

2. Give examples of plane surfaces.

3. How many planes can contain, or be passed through, one line? and what will follow if that line be perpendicular to some plane? Give examples.

4. When are planes parallel? Give examples.

5. When is a line parallel to a plane? Give examples.

**6. How many perpendiculars can be drawn to a plane from a fixed point?**

**7. A perpendicular to a plane can be perpendicular to how many lines in a plane? and how will they be situated?**

**8. How do parallel planes intersect a plane not parallel to them? Give examples.**

**9. How do the angles compare, made by equal lines from a point to a plane? What bodies present examples of such lines?**

**10. How many and what different kinds of position can two lines have? In which of them are the lines in the same plane, and in which are they not so?**

**274.** Two straight lines, or three points, not in one line, fix the position of the plane containing them; for (see Fig. 141) if the plane T were to turn about either of the lines in it, as *ab*, the instant that its motion began, it would cease to contain the other line, as *cd*. Hence, when it contains both *ab* and *cd*, or, what is the same, three points, as *a*, *n*, and *c*, its position is fixed. The face of a door opened till it strikes a stop fastened into the floor, is a plane fixed by three points; viz., the hinges and the stop.

If a line, as *mn*, is perpendicular to each of two lines, as *ab* and *cd*, Fig. 141, it is perpendicular to their plane, T, and to all other lines in that plane.

**275.** Many geometrical truths are *proved* to be true by showing that they lead back to other truths already proved to be true. But if they happen to be evident to the mind through the sight of visible examples, or if the truths employed in proving them are no simpler than these visible truths, then it is sufficient to *show* their truth by examples. Thus, any square-cornered box or room will sufficiently show that —

1st, Planes perpendicular to the same straight line (as floor and ceiling are both perpendicular to a plumb-line hung from the ceiling) are parallel to each other (see also the two flat ends of a revolving squirrel-cage, both perpendicular to the axis of the cage).

2d, A line, like the plumb-line in the last case, which is perpendicular to one of two parallel planes, is perpendicular to the other one (see also the axle of a pair of locomotive driving-wheels, which is perpendicular to the face of each wheel).

3d, If one of two parallel lines, as two plumb-lines, is perpendicular to a plane, as the floor, the other line is also perpendicular to that plane.

**Practice.**—1. How many lines or points fix the position of a plane, and what kinds of position may the lines have?

2. Mention any examples which show that planes perpendicular to the same straight line are parallel to each other.

3. Also, that a line which is perpendicular to one of two parallel planes, is perpendicular to both.

4. Also, that if one of two parallels is perpendicular to a given plane, both of them will be so.

### Angles between Planes.

**276.** In an attic, Fig. 142, where the sloping inside surface, S, of the roof meets the floor, F, in the line *ab*, and the upright end, E, of the room is perpendicular to the floor, roof, and ceiling, C, the roof-slope is said to make the angle *dac* with the floor. This angle is included between the *lines* *da*, cut from the slope S, and *ca*, cut from the floor F, by the plane E, which is perpendicular to both S and F, and therefore to their intersection, *ab*. The lines *ca* and *da* are also both perpendicular to *ab*.

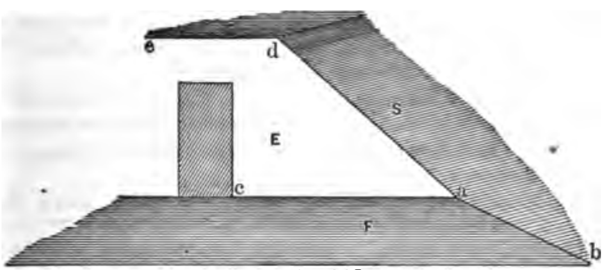



Fig. 142.

From this simple illustration, which can be seen under any roof-slope which is bounded at either end by an upright wall, we can learn several additional particulars about planes.

1st, Planes, as well as lines, make angles with each other, which may be right, as the angle between a wall and floor; acute, as that between a roof-slope and floor, or side of a tent and the ground; or obtuse, as that between a roof-slope and ceiling.

2d, These angles are called *dihedral angles*, to distinguish them from *plane angles*, or those included by two *lines;* but they are measured and represented by lines, one in each plane, and both perpendicular to the line of intersection of the two planes. Thus, again, when a door is partly open, the angle between the bottom of the *door* and the bottom of the *wall* measures the angle which the surface of the door makes with the wall.

3d, If two planes, A and B, are both perpendicular to a third plane, C, as two adjacent walls are both perpendicular to the floor, their line of intersection is also perpendicular to the plane, C (as the corner line of a room, intersection of two walls, is perpendicular to the floor).

4th, It is only expressing the last truth in a different order to add, that when a plane, A, is perpendicular to two other planes, B and C, which intersect each other (as a floor is perpendicular to the two walls which meet at the corner of a room), the plane A is perpendicular to the line of intersection of the other planes, as the floor is perpendicular to the vertical line of intersection of the walls.

**Practice.** — 1. If two sides of a yard make an acute angle with each other, what *size* and *kind* of an angle will board fences standing on those sides make with each other?

2. What is the angle between two planes called?

3. When the rafters of a roof make acute angles of *less* than 45° with an attic floor, what *size* of angles do the roof-slopes make with the floor and with each other, and what are such angles called?

4. Drive a nail at any one point on the ridge of a roof, and in how many different directions could strings be stretched from it in the surface of each roof-slope? and how many different angles could be formed between lines, one on each slope?

5. Out of all these angles, how many, and which, mark the angle which is called the angle contained between the two slopes?

6. What kind of a surface is either side of a flat board gate?

7. If such a gate were made to swing on two pins, one in the middle of the top edge of the gate, and the other in the middle of its bottom edge, what would be the position of the line joining these pins? How many different positions might the gate take in turning on these pins?

8. All the positions of the gate would represent so many positions of what, all passed through what? (See Fig. 139.)

9. If the ridge of a porch-roof is perpendicular to the front wall of a house, what position have the slopes of that roof relative to the wall?

10. If both roof-slopes of a roofed porch are perpendicular to the wall of the house, what position has the ridge of the roof relative to the wall?

11. If the front wall of a house is perpendicular to the ridge of a porch-roof, what position has it with respect to the slopes of that roof?

12. If a barrel be fitted up with sheet-iron partitions from end to end, and fastened perpendicular to the head and bottom of the barrel, would they be so perpendicular in every possible position of the barrel?

13. In how many and what positions of the barrel would they be *vertical?*

14. If one arm of a mason's or carpenter's square be applied to one edge of a floor, and the other arm to the next edge, what size and kind (or form) of angle is tested?

15. If one arm of the square be applied to the floor, and the other to the wall, what kind of angle is tested, and what position should the arms have with respect to the edge of the floor (foot of the wall)?

### Polyhedral Angles.

**277.** When three or more planes pass through one point, they form what is called a solid angle. Thus, S, Fig. 143, represents a solid angle formed of the fewest planes, viz., three; the planes S*ab* and S*bc* in front, and the plane S*ac* behind, as indicated to the eye by the dotted line in it.

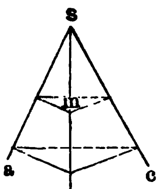


Fig. 143.

Such a solid angle is called a *trihedral*. It consists of, or embraces, six parts, — three plane angles, *a*S*b*, *b*S*c*, *a*S*c*; and three dihedral angles, one between each two of the three planes. If *m* represents the plane angle which measures the dihedral angle between the planes S*ab* and S*bc*, its vertex, *m*, is on the dividing edge, S*b*, on which those planes meet, and its sides are perpendicular to that edge at *m*. The other two dihedral angles, whose vertices are on S*a* and S*c*, are similarly placed.

If four planes meet in one point, they form a polyhedral angle, having in like manner four plane angles and four dihedral angles, as seen in a four-sided pyramid, Fig. 13. Thus, the trihedral is the simplest of polyhedral angles. The simplest trihedral is that in which all its six parts are right angles, as seen at the corner of a cube or of a square-cornered room.

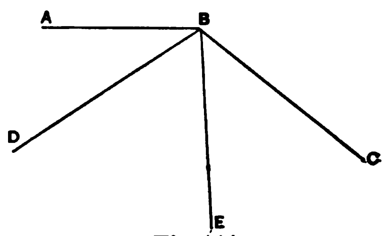


Fig. 144.

**278.** Any number of different trihedrals can easily be made by cutting out pieces of paper, and folding them with two folds, as in Fig. 144, but subject to two conditions: —

1st, The sum, ABC, of the three angles must be less than 360°, for we have seen that 360° embraces the entire space in a plane and around one point.

2d, The largest one, as DBE, of the three plane angles must be less than the sum of the other two; otherwise, when ABD and CBE were folded together on the lines BD and BE, the edges AB and CB would not meet, as desired, before reaching the surface DBE.

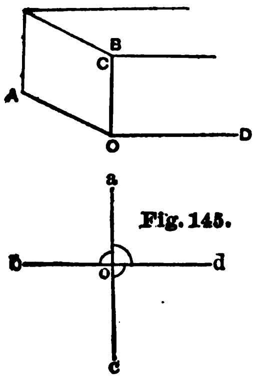



Fig. 145.

By cutting out just a quarter, *cob*, of 360°, as in Fig. 145, and folding up the right angle *aob* on *ao*, and *cod* on *do*, a trihedral right triangle, or right trihedral, as at the corner, O, of a square paper box, would be formed. *ob* and *oc* would fold together and form one line, the corner edge, O(BC), of the box (Fig. 145).

**Practice.**—1. A three-sided monument stands in a three-cornered grass plat. If its top is a regular pyramid, each angle at the vertex must be less than what?

2. A square stone gate-post is finished at top as a four-sided pyramid. Each angle at the vertex must be less than what?

3. A six-sided stone monument has a pyramidal top. Each equal angle at the vertex must be less than what?

4. Make a paper trihedral each of whose plane angles shall be 60°.

5. How many polyhedral angles are there at the summit of an octagonal spire? how many plane angles? and how many dihedral angles?

6. How many polyhedral angles at the base of the spire, and how many plane angles in each?

7. Are the plane angles and the dihedral angles at the summit of the spire acute, or obtuse?

8. Make a polyhedral angle having eight equal plane angles.

# CHAPTER VIII.

## THE ELEMENTARY BODIES.

**279.** Every geometrical body bounded by flat surfaces has sides, edges, and corners. These corners are always polyhedral angles. Having now studied plane angles and figures, and shown how polyhedrals are formed, we are ready to proceed to use all these in building up complete bodies.

*The elementary bodies* are those which are formed and bounded in the simplest manner by the straight line and circle, as already described in Chap. I. Having really been studying them in studying the lines, plane and solid angles, and plane figures out of which they are made, it only remains to describe their *forms* and varieties a little more fully, and to show how to find the *measure* of their surfaces and volumes.

**280.** Plane figures are said to be not only parallel, but similarly situated, when the *equal sides* of both lie in the same direction.

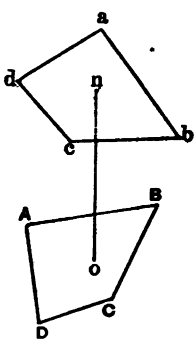


Fig. 146.

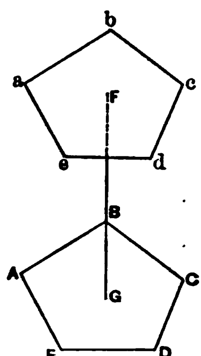


Fig. 147.

Thus, Fig. 146 represents two pieces of pasteboard of similar shape, the piece *abcd* being held over the piece ABCD, and parallel to it; while the edge, as *ab* of the piece *abcd*, is equal to the edge AB of the piece BCD, but is *not* placed parallel to it. These two figures, then, whose distance apart at all points equals *no*, are parallel, but are not similarly situated. In Fig. 147 are shown two pieces of the same shape, whose distance apart at all points equals FG, and whose equal edges, as *bc*, BC, etc., are not only equal, but parallel to each other. These figures are therefore both parallel and similarly situated.

### Prisms.

**281.** Now, to begin with the study of *Prisms*. A prism is a body two of whose faces are *equal* polygons *parallel* and *similarly situated*, and the remaining faces are parallelograms.

The equal polygons are called the bases of the prism, and the other faces taken together are called its convex surface.

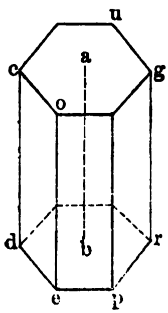


Fig. 148.

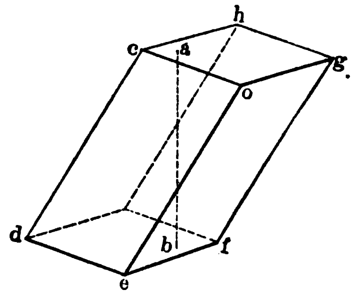


Fig. 149.

Thus, in Fig. 148, *cogu* and *depr* are the bases; *code* is a *face;* and the faces *ocde*, *oep*, *prg*, etc., all together make up the *convex surface* of the prism.

**282.** The lines, as *co* or *cd*, Figs. 148, 149, are called the *edges* of the prism, and the perpendicular distance, *ab*, between the bases is called the *altitude*. When, as in Fig. 148, the edges, as *cd*, *oe*, etc., are equal and parallel to the altitude, the prism is said to be a *right* prism; but when, as shown in Fig. 149, these edges are not parallel to the altitude *ab*, the prism is said to be an *oblique* prism, and the edges, as *cd*, *oe*, etc., are longer than the altitude *ab*.

**283.** Then, again, as to the number of sides in each base: if the base is a triangle, the prism is called *triangular;* a *quadran-*

*gular* prism is one whose bases are quadrilaterals; a *pentagonal* prism is one whose bases are pentagons; a *hexagonal* prism is one whose bases are hexagons; and so on. And, once more, when the bases are regular, the prisms are called regular; thus, if the bases are equilateral triangles, the prism is a *regular* triangular prism; and if the bases are regular hexagons, the prism is a *regular* hexagonal prism, etc.

[Models of these should be shown.]

**284.** All *quadrangular* prisms of which the bases are *parallelograms*, are called *parallelopipedons*. Fig. 149 represents a prism of this sort.

Recollecting that a *rectangle* is only a particular kind of *parallelogram*, all *parallelopipedons* of which the faces are *rectangles* are called *rectangular parallelopipedons;* and recollecting, again, that a square is only a particular kind of rectangle, the *cube* is merely that kind of *rectangular parallelopipedon* of which all the faces are *squares*.

**285.** Let us now review the foregoing statements, putting them in the form of a table, thus: —

I. Prisms, with respect to the *relative position of their altitudes and edges*, may be
  (*a*) Right prisms, or
  (*b*) Oblique prisms.

II. Prisms, with respect to the *shape of their bases*, may be
  (*a*) Regular prisms, or
  (*b*) Irregular prisms.

III. Prisms, with respect to the *number of sides in their bases*, may be
  (*a*) Triangular prisms.
  (*b*) Quadrangular prisms.
    1°. That *kind* of *quadrangular prism* of which the bases are parallelograms, is a *parallelopipedon*.
    2°. That *kind* of *parallelopipedon* of which the bases are rectangles, is a *rectangular parallelopipedon*.
    3°. That *kind* of *rectangular parallelopipedon* of which the faces are squares, is a *cube*.
  (*c*) Pentagonal prisms.
  (*d*) Hexagonal prisms, etc.

**286.** *Regular right prisms* are oftenest seen. The representation of these on paper, exactly or in pictorial diagrams, has been mentioned in Chap. II.; but it will be more useful here to show how to form patterns of their unfolded surfaces, from which paper models of them can be formed

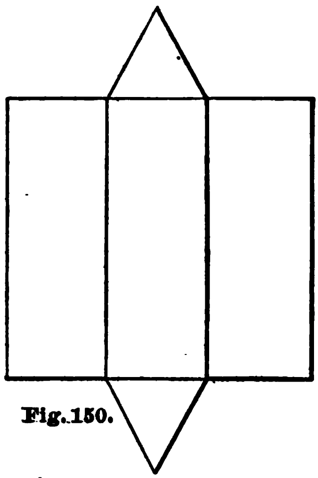


Fig. 150.

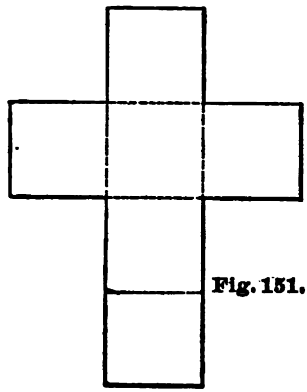


Fig. 151.

Fig. 150 is the pattern of the convex surface and bases; that is, of the total surface of a triangular regular right prism. Fig. 151 is the pattern of the entire surface of a cube, the dotted lines

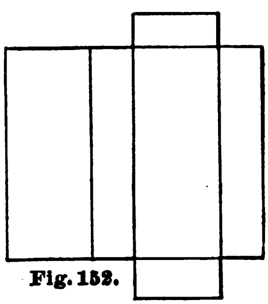


Fig. 152.

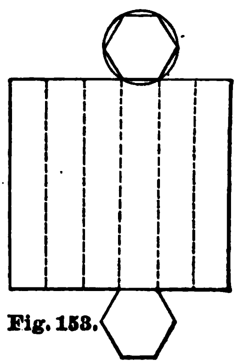


Fig. 153.

showing where the pattern, when cut out whole, should be folded so as to form a cube. Fig. 152 is the pattern of a rectangular parallelopipedon, and Fig. 153 is that of a regular hexagonal prism.

**287.** These patterns of prisms make it easy to see that the total area of the surface of a prism is the area of the convex surface added to that of the bases. Moreover, when the convex surface of a right prism is unfolded, it is seen to form a rectangle whose base equals the perimeter of the base of the prism, and whose altitude is the altitude of the prism. See, then, Chap. VI. for the rules for finding the areas of rectangles and polygons.

**Practice.** — 1. Find the total surface of a square prism each of whose sides is $2\frac{1}{2}$ feet wide and $8\frac{3}{4}$ feet high.

2. Find the total surface of a cube whose edge is 1 inch.

3. Find the total surface of a cube whose edge is 6 feet.

4. How much cloth will cover the convex surface only of a marble-topped hexagonal stand, each side of which is 11 inches wide and 32 inches high?

5. How much paper will paper the walls and ceiling of an octagonal apartment, each side of which is 5 feet wide and 9 feet high, and with a door 2 feet by 7 feet 3 inches in one side, and a band of window openings 2 feet 8 inches high all around the rest of the wall-space?

**288.** By the solidity or volume of a body, we mean the number of solid, or cubic, feet, etc., contained in that body. In the prism, Fig. 154, let the divisions represent feet. We know already that the number of square feet in the base, HDEF, equals the length, DE, multiplied by the width, EF. Now, by cutting off one foot of the height of the prism, there will be a *layer* of solid feet, HDEF*adcb*, which will contain as many solid feet as there are square feet in the base, and there will be as many such layers as there are feet in the height of the prism. Hence the

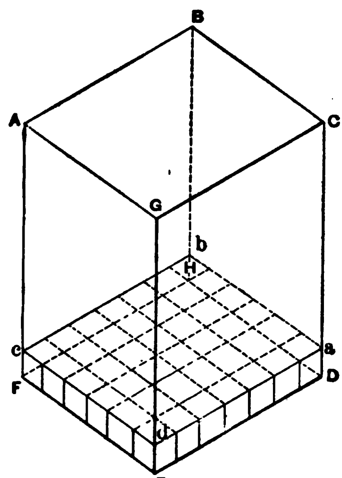


Fig. 154.

Rule. — *To find the solidity of a prism, multiply its length by its breadth, and the product by the thickness;* or, calling one of the ends of the prism its base, *multiply the area of the base by the altitude.*

**289.** When the length, breadth, and thickness of a prism are equal, we have the following

Rule. — *To find the solidity of a cube, multiply the length of its edge twice by itself.*

Thus, if the edge of a cube is 2 inches, its solidity is 2 × 2 × 2 = 8 solid, or cubic, inches.

Since multiplying a number by itself twice gives the solidity of a cube whose edge is represented by that number, the multiplication is called cubing the number, and the product is called the cube of the number.

**Practice.** — 1. What is the solidity of a prism whose length is 10 feet, width 4 feet, and thickness 2 feet?

2. What is the solidity of a cube whose edge is 5 feet 5 inches?

3. What is the capacity of a box, without a cover, whose outside dimensions are as follows: the length 14 inches, the width 14 inches, and the height 13 inches; the box being made of inch boards?

4. How many cords of wood in a pile 200 feet long, 20 feet wide, and 8 feet high?

5. What is the capacity of a cistern 8 feet square inside, and 6½ feet deep? and how many gallons will it hold at 231 cubic inches to a gallon?

6. How many horse-loads of earth at 32 cubic feet to a load can be put into a pen 7 feet square and 5 feet 3 inches deep?

**290.** Let ABCD, EF, Figs. 155 and 156, be two prisms having equal squares, ABCD, for their bases, and equal heights, AF =

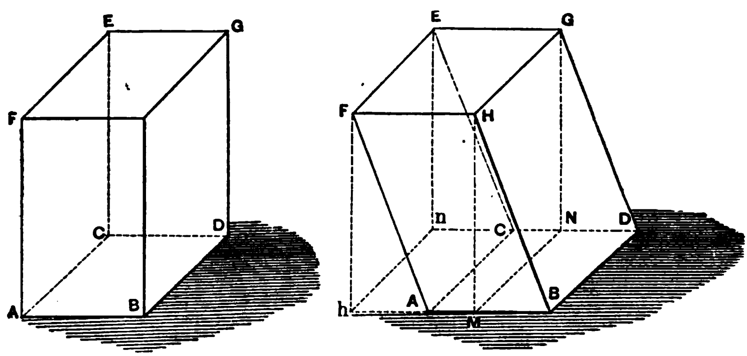



**Fig. 155.** **Fig. 156.**

*h*F. Then the solidities of these prisms are equal. For, suppose the first prism, Fig. 155, to be composed of many thousands of sheets of very thin paper pressed so tightly together as to make the convex surface of the prism smooth and hard. Next suppose each of these sheets to be slipped a very little — too little to be seen — to the left of the one below it, beginning with the one next to the bottom one, so that they shall finally form the second prism, Fig. 156. There are equal amounts of paper in the two prisms, and hence they have equal volumes, or solidities.

A more purely geometrical manner of looking at this kind of proof is as follows: —

Let the base ABCD be of a thickness too *small* to think. Then the number of positions, touching each other, which it will take in moving directly up to the position FEG, will be too *great* to think. Now, the number of these positions in both prisms will be the same, whatever it be, because the height of the prisms is the same. But the sum of all these positions makes the solid contents of the prism; hence this solid contents is the same for both prisms.

NOTE. — The learner should take pains to think this proof, for, *first*, it is a very simple example of one of the most beautiful forms of thinking that geometry teaches; and, *second*, the advantage of this method is, that it applies to all right and oblique bodies of the same kind, and having the same base and height, — to cylinders, cones, and pyramids, as well as to prisms of every form.

**291.** *Second Proof.* — By cutting off from the oblique prism ABCD, EFGH, Fig. 156, the triangular prism MBDN, HG, by a vertical plane through HG, and putting the prism thus cut off at ACnh, EF, the oblique prism will be transformed into the right prism MNhnEFGH, which will evidently be equal to the oblique prism.

The result would have been the same if the thin sheets had each slid a very small distance in a *diagonal* direction, as AD, instead of in a direction parallel to a side, as AB, of the base. Hence, finally, to find the solid contents of any prism, right or oblique, we have the

RULE. — ***Multiply the area of the base by the perpendicular height or altitude.***

**Practice.** — 1. A pile of boards is 16 feet square at every point in its height, and is 20 feet high. It leans by the overlapping of each layer, until at the top it is 3 feet out of a vertical direction. Are there more or less boards, or quantity of lumber, in it than if its sides were truly vertical? and how many cubic feet are there in the pile?

2. Having a block 6 inches square and 20 inches long, I sawed a piece off obliquely at one end, and fastened it to the other end, so as to make an oblique quadrangular prism. Calling the sawdust nothing, was there any more or less wood than at first? and how many cubic inches were there in the block?

3. A wall which holds a bank of earth is 80 feet long, 12 feet high, 6 feet thick at base, and 4 feet thick at top. How many cubic feet are there in each foot of its length, and how many cubic yards in the whole of it?

4. A section of straight railroad embankment across a level meadow is 12 feet wide at top, 36 feet wide at bottom, and 8 feet high. How many cubic yards in 100 feet of its length?

### Cylinders.

**292.** The cylinder (47) being only a prism of an infinite number of plane sides or faces, as the circle, its base, is a polygon of an infinite number of sides (170), its surface and volume are found in the same way that those of prisms are.

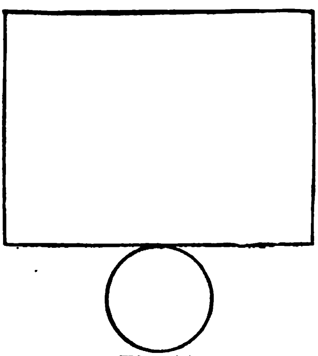


Fig. 157.

Indeed, when a cylinder, as a joint of stove-pipe, is cut open and rolled out flat, it forms a rectangle, one side of which is equal to the circumference of the cylinder, and the next side is equal to the height of the cylinder (see Fig. 157).

**293.** Strictly speaking, a body is a cylinder which has *any* curve for a base, and a convex surface on which parallel straight lines can be drawn in any *one* direction (49); but here we shall only consider the right cylinder with a circular base, also called a cylinder of revolution, because it can be formed by revolving a rectangle about one of its sides as a fixed axis. Thus, when a door or blind swings on its hinges, its outer edge generates the convex surface of a cylinder of revolution. Accordingly we have the following

RULES. — I. *To find the convex surface of a cylinder, multiply the circumference of its base by its altitude.*

II. *To find its total area, add to the convex surface the area of the two bases.*

III. *To find the solid contents of a cylinder, multiply the area of its base by its altitude.*

**Practice.**—1. What is the area of the convex surface of a cylindrical block 4 inches in diameter and 8 inches long?

2. How many square feet of sheet-iron will it take to make 40 feet of 6½-inch stove-pipe?

3. How many square feet of tin will it take to make 6 water-conductors, each 20 feet long and 3 inches diameter?

4. What is the area of the convex surface and bottom of a pail 1 foot in diameter and 16 inches high?

5. If a pail, to contain 60 cubic inches, is to be 4½ inches in diameter, how high must it be?

6. If a sheet of tin 10 inches by 18 inches is rolled up into a cylinder 10 inches long, what will be the diameter of the cylinder?

7. A locomotive cylinder is 17 inches inside diameter, and its piston makes a stroke of 22 inches, is 3 inches thick, and comes to ½ inch from the ends of the cylinder at the end of its stroke. What is the interior capacity of the cylinder?

**294** Having before shown (56, 57) that the cone is the general form of which the cylinder is a particular case, it is natural next to consider the surface and volume of the cone.

### Cones.

The cone, in the broad sense, may, like the cylinder, have *any* curve for its base, and its convex surface may then consist of straight lines joining all points of that curve with *any one* point out of the curve, and out of its plane if it be a plane curve; but we shall here consider only the simplest possible cone,—the common cone with a *circle* for its base, its vertex on a perpendicular to the base at its centre, and hence having its *slant height* everywhere the same, and its *altitude*, or perpendicular, from the vertex to the base, coinciding with its *axis*, or the line from the vertex to the centre of the base.

**295.** The convex surface of a cone, when rolled out flat, will be the *sector of a circle* whose radius will be equal to the slant height, and whose arc will equal the circumference of the base of the cone.

Thus, place the cone upon a table, as shown in Fig. 158, mark that point, *a*, of the base which touches the table, and then roll the cone, without letting it slip, till the point *a* is again found in the surface of the table, when it will be seen that the sectoral space thus traced out upon the table will be equal to the cardboard sector which just covers the convex surface of the cone.

The vertex, V, cannot roll at all, it being only a point, and therefore having no diameter; hence it remains stationary at the centre of the sector rolled over by the surface of the cone.

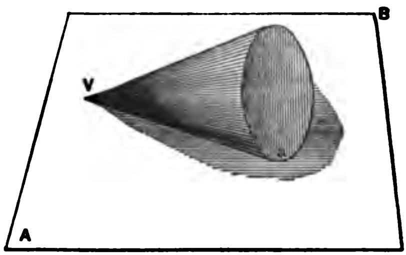


Fig. 158.

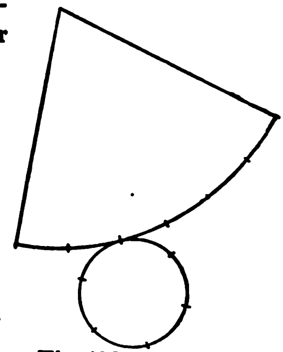


Fig. 159.

Fig. 159 shows the base and unfolded convex surface of a cone whose base is half an inch in diameter, and whose slant height is one and a quarter inches.

From the foregoing and (264, 265), the convex and total surface of a cone is found by the following

RULE. — *Multiply the circumference of the base by one-half the slant height for the convex surface, and add the area of the base to find the total surface.*

**Practice.**—1. What is the area of the convex surface of a conical tower roof 12 feet in diameter at base and 15 feet high?

2. The flattened conical cover of a fountain is 15 feet in diameter and 3 feet high. What is the slant height and area of the conical surface?

3. How many square yards of canvas in the walls and floor of a conical tent 13 feet in diameter and 15 feet high?

**296.** *The Volume or Solid Contents of the Cone.* — The idea of motion can often be introduced into geometrical reasoning with great advantage. It also naturally and properly may be; since, as we have already often seen, geometrical forms can be defined as the result of motion, as when an angle is defined as the result of a partial revolution of a line about a fixed point of the line, or a sphere as formed by the revolution of a semicircle about its diameter (59).

If, for example, a level square of 3 inches on each side should move 6 inches in a direction perpendicular to its surface, it would generate a right square prism 6 inches high, whose solid contents would be $3 \times 3 \times 6 = 54$ cubic inches = the area of the square multiplied by the distance through which it had moved.

Now, if the square had moved in such a manner that its final position would not be parallel to its first position, all its points would not have moved the same distance, and the solid contents of the body generated by it would be the area of the square multiplied by the *average distance* travelled by all its points; that is, the distance which they would all have moved if they had all travelled the same distance, and a body had been formed equal to the given body.

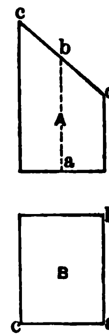


Fig. 160.

For example, if *ab*, the *average* of the different heights of the square-bottomed tin box A, Fig. 160, be equal to the uniform height, *ab*, of the box B, and if the boxes are of the same size square, they will be equal in volume, as may be seen by their holding equal quantities of water if A is inclined until its top, *cd*, becomes level.

These principles of motion enable us to prove very simply and beautifully the following theorem, which is otherwise somewhat difficult: —

**297. Theorem.** — *The solid contents of a cone is one-third of that of a cylinder of the same base and height.*

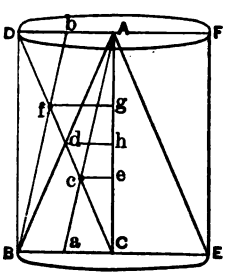


Fig. 161.

The cylinder BDEF, Fig. 161, is generated by the revolution of the rectangle ADBC about the side AC. Its volume, by the principles just explained, will equal the area of the rectangle multiplied by the circle described by its middle point, *d*.

The cone, ABE, is generated by the revolution of the triangle ABC about the side AC. Its volume is therefore equal to the area of this triangle multiplied by the average of the distances travelled by all its points in one revolution. Now, *a* being the middle point of BC, the line A*a* divides the triangle ABC into the triangles AB*a* and AC*a*, which are equivalent, because they have equal bases and the same height, AC. Hence the line A*a* represents the average distance of all points of ABC from AC. Again, as *d* is the middle point of AB, the line C*d* divides the triangle into the *halves* AC*d* and BC*d*, having the equal bases A*d* and B*d*, and for the common altitude a perpendicular from C to AB. Hence the line C*d* also represents the average distance of ABC from AC. Therefore the point *c*, intersection of A*a* and C*d*, is the *point* whose distance, *ce*, from AC is the average distance of all the points of the triangle from AC. Hence the solid contents of the cone is equal to the area of the triangle ABC × the circle described by the point *c*. But ABC = $\frac{1}{2}$ ACBD, and *ec* = $\frac{2}{3}$ *dh*, since the parallels, A*a* and B*b*, evidently divide CD into thirds, and the parallels *fg*, *dh*, and *ce*, to AD, give C*e* = $\frac{1}{3}$ CA = $\frac{2}{3}$ of $\frac{1}{2}$ CA = $\frac{2}{3}$ of C*h*, and hence *ce* = $\frac{2}{3}$ *dh*.

Hence ABC × circle *c* = $\frac{1}{2}$ ABCD × $\frac{2}{3}$ circle *d* = ABCD × $\frac{1}{3}$ circle *d*; that is, the volume of the cone is one-third of that of the cylinder. Hence, as the volume of a cylinder is commonly found as the product of its base and altitude, that of the cone is its *base* × $\frac{1}{3}$ *of its altitude*.

**Practice.**—1. Find the volume of a cone 6 inches high and 3 inches diameter.

2. Find the volume of a cone whose height is 3 feet, and radius of base 16 inches.

3. A cone is designed to be 8 feet diameter at base, and 80 feet high; 9 feet in height from the base is made in brick. What is the volume of the whole, and of the part constructed?

4. The slant height of a cone is 16 feet, and altitude 11 feet. What is its volume?

5. The radius of a cone's base is 7 feet, and altitude 9 feet. What is the volume?

6. The diameter of a cone's base is 12 feet, and slant height 10 feet. What is its volume?

**298.** If the upper part of a cone be cut off by a plane parallel to its base, the remainder is called a frustum (*pl.* frusta). Frusta are oftener seen in practice than complete cones are, hence it becomes important to be able to find their surfaces and solidities.

For finding the area of the convex surface of a frustum, we make use of the rule for finding that of a trapezoid. For, let V – ABCD, Fig. 162, be the unrolled surface (development) of a conical surface, and AD*ad* that of a frustum of that surface divided into equal sections, AB$ab$, BC$bc$, etc. When these sections are exceedingly many and small, each becomes a trapezoid whose area $= \frac{1}{2}(AB + ab) \times h$. Hence the entire convex surface AD$ad$ of the frustum $= \frac{1}{2}(AD + ad) \times h$, where $h$ is the slant height of the frustum.

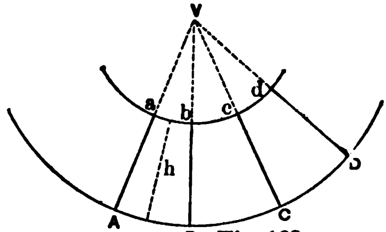


Fig. 162.

**Practice.** — 1. How much tin will make the conical taper of a can, its upper base being 6 inches diameter, its lower base 9 inches, and its altitude 4 inches?

2. What is the convex and total surface of a conical wooden cistern having a slant height of 8 feet, a bottom diameter of 9 feet, and top diameter of $6\frac{1}{2}$ feet?

3. Find the convex surface of a copper reservoir 14 inches bottom diameter, 10 inches top diameter, and 3 feet 6 inches high.

4. Find the convex surface of a conical glass measure 15 centimeters high, 8 centimeters radius of lower base, and 5 centimeters radius of upper base.

5. What is the area of a conical tub and cover, its lower base 0.75 meter in diameter, upper base 0.60 meter diameter, and slant height 0.48 meter?

**299.** The direct rule for finding the volume of a frustum is too complicated to be explained here; but, fortunately, there is a very simple indirect one, as follows: —

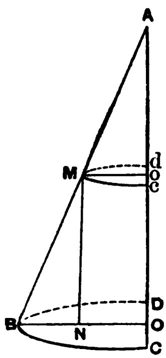


Fig. 163.

Let A – BCD, Fig. 163, be a half-cone, of which M$cd$ – BCD is a half-frustum. Suppose the radius OB = 12 inches, M$o$ = 7 inches, and altitude O$o$ (= MN) = 15 inches, are given. Then, evidently, the triangles MBN and ABO are similar (154), and BN is the same part of MN that BO is of OA; that is,

$$\frac{MN}{BN} = \frac{AO}{BO}, \quad \text{or } \frac{15}{5} = 3 = \frac{AO}{BO} = \frac{AO}{12},$$

or AO = 3 × 12 = 36 inches.

We can now find the volume of the entire cone ABO

$$= \pi \times BO^2 \times \tfrac{1}{3} AO = 3.1416 \times 144 \times 12.$$

Also, the volume of the upper cone AM$o$

$$= \pi \times Mo^2 \times \tfrac{1}{3} Ao = 3.1416 \times 49 \times 7.$$

Then, by subtracting the latter volume from the former, we shall have the volume of the frustum.

Otherwise, if the slant height MB were given, the equation

$$\frac{MB}{BN} = \frac{AB}{BO}$$

would give AB; and then we could find A, knowing that

$$AO = \sqrt{AB^2 - BO^2},$$

also

$$MN = \sqrt{MB^2 - BN^2}.$$

**Practice.**—1. How many cubic feet in a log 5 feet in diameter at one end, 3 feet 8 inches in diameter at the other end, and 16 feet long?

2. How many cubic feet in a cistern whose bottom diameter, inside, is 9 feet, top diameter 7½ feet, and depth 8 feet, and how many gallons, of 231 cubic inches each, will it hold?

3. A paper pail is 11 inches inside diameter at bottom, and 14 inches at top, and is 12 inches deep. How much water is in it when filled up to 2½ inches from the top?

4. A furnace-stack is 8 feet diameter at base and 6 feet at top, is 20 feet high, and has a cylindrical interior 4 feet in diameter. How many bricks, each containing 64 cubic inches, will be required to make it?

5. A tapering cast-iron shaft is 6 feet long, 15 inches diameter at one end, and 11 inches at the other. What will it weigh if 7¼ times as heavy as water at 62½ pounds per cubic foot?

### The Pyramid.

**300.** The cone is related to the cylinder on the one hand, the latter being that particular case of the cone in which the axis is of indefinitely great length (56). It is related, on the other hand, to the pyramid, since both have a vertex at a finite distance from the base, and the cone is that particular pyramid in which the number of sides is infinite. We therefore find it convenient to consider here the surface and volume of a pyramid and its frustum.

If we take any polygon, and make its sides the bases of a set of triangles which shall all have a common vertex at some point *not in the surface containing the polygon*, the body, bounded by the polygon and the triangles, will be a pyramid.

[This may be illustrated by reference to models of pyramids.]

The polygon is called the *base* of the pyramid, and the triangular faces together form its *convex surface.*

As in the case of prisms, the boundaries of the different faces are called *edges.*

The point where the triangular faces all meet is called the apex, or *vertex*, and the perpendicular distance from the vertex to the base is the *altitude.*

**301.** When the base is a regular polygon, and the altitude passes through its centre, the pyramid is a *regular right pyramid.* In the pyramid shown in Fig. 164, *abcd* is the base, *abe* — *ebc* — etc. make up the convex surface, *eo* is the altitude, *e* is the vertex, and *er* or *ep*, etc., drawn from the vertex perpendicular to the edges, *ab* or *bc*, etc., of the base, is called the *slant height* of the pyramid.

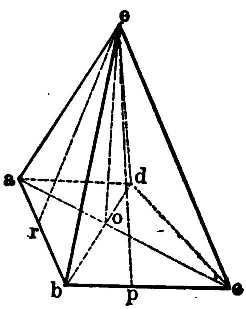


Fig. 164.

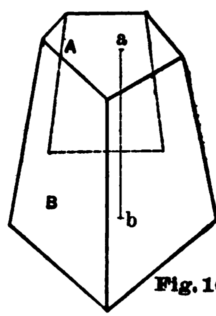


Fig. 165.

If a part of a pyramid be cut off so that the bases A and B, Fig. 165, of the remaining part are parallel, this remaining part is called a *frustum* of the pyramid, and its *altitude*, *ab*, is the perpendicular distance between its bases, A and B.

Again, as in the case of the prism, a pyramid is said to be triangular, quadrangular, etc., according as the base is a triangle, quadrangle, etc.

**302.** *The Surface of the Pyramid.* — The convex surface of a pyramid is made up of triangles. We therefore have only to find the areas of these triangles, and to add them, in order to obtain the area of the convex surface of the pyramid.

If the base be a *regular* figure, and the pyramid a *right* one, the triangular faces will be equal, and the area of one of them multiplied by the number of them will determine the area of all of them.

The common altitude of the equal faces of a regular pyramid is the *slant height*, or line drawn from the vertex, perpendicular to any side of the regular polygon which forms the base. Therefore the

RULE. — *Multiply the perimeter of the base by the slant height.*

**Practice.**—1. What is the *convex* surface of a pentagonal pyramid in which each side of the base is 5 feet 4 inches, and the slant height 7 feet 5 inches?

2. What is the *total* surface of a pyramid with a square base, each of whose sides is 3 inches, and whose slant height is 11 inches?

**303.** *The Volume of any Pyramid* is one-third of that of the prism having the same base and altitude, for (see Figs. 166 and 167) the triangles VOG and VHG are equal (Fig. 166). But VOG generates the pyramid V – ABMN by turning about VO, and with its centre *e* describing a square, of which *ge* is half of one side; the triangle varying in size as it turns.

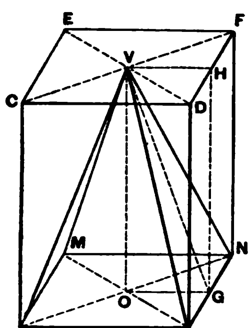



Fig. 166.

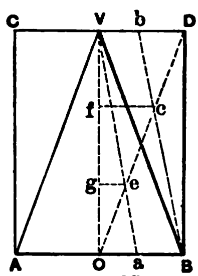



Fig. 167.

The *equal* and *equally* varying triangle VHG (see VOA = VCA, Fig. 166) generates the portion of the prism ABMN – CF outside of the pyramid; its centre *c*, Fig. 167, describes the square half of whose side is *fc*, which is evidently equal to 2*ge*; that is, *c* travels twice as far as *e*. Hence the pyramid generated by VOG, being one-half of the solid generated by VHG, is one-third of the whole prism which is generated by both triangles moving together.

Like reasoning applies to any pyramid. Hence, to find the volume of any pyramid, we have the

RULE. — *Multiply the area of its base by one-third of its altitude.*

The volume of the frustum of a pyramid is found in the same way as that of a cone.

**Practice.** — 1. How many cubic feet in a square pyramid 4 feet square at base, and 12 feet high?

2. A bin 6 feet square at top, and 4 feet square at base, and 5 feet deep, contains how many bushels at 2,150.42 cubic inches each? How many gallons at 231 cubic inches each? How many liters at 35.24 liters to 1 bushel? How many cubic meters at 0.0283 cubic meter to 1 cubic foot?

3. What kind of triangles are the faces of the convex surface of any regular pyramid?

4. Each side of the base of a regular triangular pyramid is 3 feet, and its altitude 8 feet. How many cubic yards in it?

5. Each side of an octagonal pyramid is 1.4 meters, and its altitude is 4 meters. How many cubic meters in it?

6. A square frustum is 6 feet square at base, 4 feet square at top, 16 feet high, and has a hollow interior 3 feet square. How many cubic feet are there in it?

7. How many cubic inches in the largest octagonal pyramid that can be cut from a block 2 feet square and 1 foot thick, and what part are they of the cubic inches in the block?

### The Sphere.

**304.** *A sphere* is a body all points of whose surface are equally distant from a point within called the centre. *A segment* of a sphere is a portion of that body having two parallel bases, which are circles on the surface of the sphere, and having for its convex surface that portion of the spherical surface which is included between these bases. Or it may be bounded by only one circular base, together with a portion of the surface of the sphere. In the former case the segment is called a segment of two bases, and in the latter a segment of one base. When the sphere is cut through its centre, the two segments are equal, and each is called a *hemisphere.*

The *sector* of a sphere is a solid bounded by a conical surface having the centre of the sphere for its vertex, together with a portion of the spherical surface for a base. [These can all be cut from a round apple or potato, or shown in models, and their shape may thus be made evident.]

Thus, in Fig. 168, *fghk* – *mnop* is a segment of two bases; *mnop* – *s*, or *abcd* – *e*, is a segment of one base; *fghk* – *e* and *fghk* – *s* are hemispheres; and *abcd* – *e* – O is a spherical sector. *fh* is a diameter, and O*h*, O*e*, etc., are radii of the sphere.

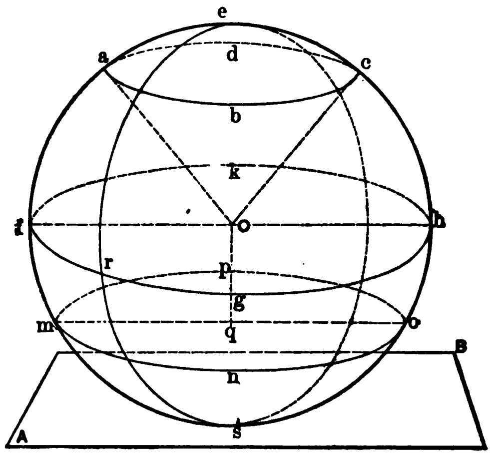



Fig. 168.

**305.** Proceeding to the consideration of the parts of the *surface* of the sphere, a *zone* is a portion of the surface of a sphere included between two parallel circular bases; or, it is the base of a sector. Thus, the surface between *fghk* and *mnop* is a zone having two bases, and the surface *abcd* – *e* is a zone having but one base; viz., the circle *abcd*. The torrid and temperate zones of the earth have two bases. The frigid zones are zones of one base.

*The altitude* of a zone or of a segment is the perpendicular distance between the parallel circles which form the bases of the zone or segment; thus, O*q* is the altitude of the segment *fghk* – *mnop*, and of the zone which bounds it.

A plane is said to be *tangent* to a sphere when it touches the surface in only one point; thus, AB represents a horizontal plane tangent to the sphere (Fig. 168) at its lowest point, and on which the sphere therefore rests.

**306.** It follows from the definition of a sphere, that, if it is cut through in any direction, the *section of it will be a circle*. All circles formed by cutting a sphere through its centre are called *great circles;* all other sections of a sphere are *small circles*. Or, in other words, a great circle is one whose diameter passes through the centre of the sphere, and a small circle is one whose diameter does not pass through that centre. The diameter of any small circle is a *chord* of the sphere. Thus (see Fig. 168), *mnop* is a small circle, and *mo*, its diameter, is a chord of the sphere; *fghk* is a great circle, and *fh*, its diameter, is also a diameter of the sphere.

When any one great circle, as *fghk*, Fig. 168, is chosen for the purpose of comparing others with it, the small circles parallel to it, as *mno*, are called *parallels* of the sphere, while all the circles which cross it at right angles, as *efs* or *ers*, are called *meridians*.

The parallels of latitude and meridians of longitude, which we learn about in geography, are imaginary lines on the earth, which is a sphere, and are shown on globes, and compared with the fixed great circle, which is called the equator.

If three great circles are drawn upon the surface of a sphere so as to intersect each other, as in Fig. 169, the three-sided figures, *abc* and *def*, formed by their intersection are called *spherical triangles*. (In the figure, the circle *mno* is supposed to divide the part of the sphere which is seen from the part which is out of sight, and so the dotted lines represent those parts of the three great circles which are not seen, and the full lines represent those parts of the same great circles which are seen.) [A model of a sphere, with the various lines which have been described traced upon it, should be used in illustrating the statements of this chapter.]

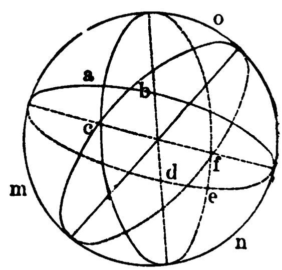


Fig. 169.

**307.** If we should take a wooden ball with dull black or slated surface in one hand, and a crayon or red lead-pencil in the other, how should we draw a circle, either great or small, on the spherical surface? It cannot be done as easily by the unaided hand on a sphere as on a flat surface, since the absence of flatness on a sphere leaves the hand to wander hither and thither, forming a curve which would not lie in a plane, as a circle does.

*One way* of drawing a circle on a sphere would be, to cut any circle, not larger than a great circle (306) of the given sphere, out of a card, and then, setting the sphere into the circular opening thus made in the card, draw the circle by following the line where the edge of the opening joins the surface of the sphere.

*Another way*, and a better one, is, to take a pair of compasses having both of its legs jointed, so that, when bent, they will conveniently straddle the sphere, as if with one point at $a$, and the other at $c$, Fig. 168. Then, for example, having set the steel point of the compasses at $e$ for a centre, describe the circle $abcd$, just as in drawing a circle on a flat surface.

*To draw a great circle*, as $fghk$, Fig. 168, remember that $ef$, the radius with which to draw it, is equal to $\sqrt{Oe^2 + Of^2}$. But, as $Oe = Of$, we have $ef = \sqrt{2Oe^2} = Oe\sqrt{2} = 1.414Oe$.

**Practice.**—1. Having given any kind of a sphere or ball of wood, tin, leather, copper, iron, or stone, on which a line can be pencilled or scratched, mark one or more circles upon it by means of cards having suitable circular holes cut in them.

2. Draw two or more parallel small circles on a sphere. [The centre, as $e$, for drawing such circles, is called their *pole*.]

3. Draw one or more great circles perpendicular to a given great circle. [The circumferences of the required circles will pass through the poles of the given circle.]

4. Draw several intersecting great circles, and note the spherical triangles (Fig. 169) which they will form by their intersections with each other.

5. Draw, as accurately as you can, circles representing the equator, tropics, and polar circles.

### The Surface and Volume of the Sphere.

**308.** In order to find the surface of a sphere, we will first explain a simple change in the rule (298) for finding the convex surface of a frustum of a cone. Let ABCD, Fig. 170, be a half-frustum. Its convex surface = AB × half the sum of the circumferences of its bases $= AB \times \frac{2\pi BC + 2\pi AD}{2}$; or, calling $BC = r$, and $AD = R$, $= AB \times 2\pi\left(\frac{R + r}{2}\right) = AB \times 2\pi MN$.

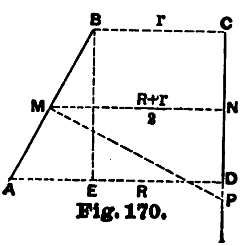


Fig. 170.

BE being a perpendicular to AD, if the triangle ABE be turned 90° about the point B, all the sides will be perpendicular to their present position. But the sides will then all have the same direction as those of the triangle MNP, MP being drawn perpendicular to AB.

Hence the triangles ABE and MNP are similar, and MN is the same part of MP that BE is of AB; that is,

$$\frac{MP}{MN} = \frac{AB}{BE},$$

hence

$$MP \times BE = AB \times MN.$$

Multiply both sides of this equation by $2\pi$, and we have

$$2\pi MP \times BE = 2\pi MN \times AB.$$

The second member of this equation expresses the usual rule for finding the area of the convex surface of a frustum. The first member, being equal to the second, shows that we can equally well use the following

RULE.—*Multiply the altitude of the frustum by the circumference (2πMP) of a circle whose radius is perpendicular to the slant height at its middle point, and limited by the axis of the frustum.*

Now, to apply these principles to finding the surface of a sphere: let the sphere AEBF be divided by parallel circles into any number of zones, equal or unequal, as GHEF, MNB, etc.

In each zone inscribe a frustum (or cone) as the one whose slant height is GE in GHEF, and the one whose slant height is MB in MNB. Then the convex surface of the frustum GHEF = OP × 2πOC, that of the cone MBN = BQ × 2πOD, etc.

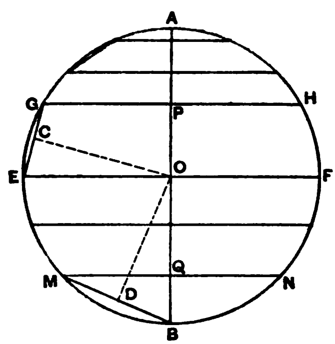


Fig. 171.

Now, when the zones are reduced to lines, the sum of the lines BM, EG, etc., becomes the semicircle AEB; the sum of the altitudes, BQ, OP, etc., equals the diameter AB; and the radii, OC, OD, etc., become equal to OE = $r$ (the radius of the sphere). Hence the surface of the sphere becomes AB × 2πOE, or (as AB = $2r$) = $4\pi r^2$.

But $\pi r^2$ is the area of a great circle of the sphere (258); hence $4\pi r^2$, the surface of the entire sphere, is equal to four times the area of one of its great circles.

The same reasoning applied to any zone, as GHEF, gives for its surface $\pi r \times$ OP, or the circumference of a great circle multiplied by the height of the zone.

**Practice.** — 1. How many square inches in the surface of a toy balloon 8 inches in diameter?

2. How many square feet of gilding in a gilded ball 2 feet in diameter?

3. Compare the cost of gilding a cylinder, and a sphere whose diameter is the same as the diameter and altitude of the cylinder, at the same price per square inch.

4. How many square meters are there in the surface of a hemispherical dome 12 meters in diameter?

5. How much leather will cover a football 18 centimeters in diameter?

6. How much leather will cover a football 10 inches in diameter?

7. What is the area of a zone of the earth's surface around the pole whose height is 1,000 miles?

**309.** The sum of the solid contents of any number of pyramids having equal altitudes is equal to the sum of their bases multiplied by one-third of the common altitude. Then, if we suppose the parallels and the meridians of a globe each to be numberless, they will divide the surface of the globe into the bases of numberless pyramids, all having the centre of the globe for their common vertex, and its radius for their common altitude. Hence the volume of a sphere equals *the product of its surface by one-third of its radius.* Or,

$$V = 4\pi r^2 \times \frac{r}{3} = \tfrac{4}{3}\pi r^3;$$

or, as $r = \frac{d}{2}$, $d$ being the diameter of the sphere, $r^3 = \frac{d^3}{8}$, and the solidity of the sphere

$$V = \frac{\pi}{6} \times d^3.$$

**Practice.** — 1. Find the volume of a sphere of 5 centimeters diameter.

2. What is the volume of a sphere 16 inches in diameter?

3. The largest sphere that can be turned from a wooden cube of 1 cubic foot contains what part of a cubic foot?

4. A lead prism of 40 cubic inches is melted into a ball. What is the radius of the ball?

5. An iron ball placed in a vessel full of water displaces 1,600 cubic inches of water. What is the radius of the sphere?

6. The volume of a sphere whose radius is 1 foot, is what part of that whose radius is 2 feet?

**310.** Notice, that, as $r$ (or $d$) is the only quantity that is changeable in the expression for the volume of a sphere, *the volume of a sphere varies as the cube of its radius or diameter.* This means, that if one sphere has two times the radius of another, its volume will be the cube of 2, or eight times that of the other.

The same principle is true for any two bodies which are of the same form.

### Truths of Form and Position.

**311.** Besides the truths of *measure* relating to the elementary solids, there are some simple, interesting, and useful ones of *form* and *position*, among which are the following: —

If a cube be cut by a plane (as a broad hand-saw) through the three corners that are equidistant from a given corner, the section will be an *equilateral triangle*, each side of which will be the diagonal of a face. (See ADE, Fig. 110.)

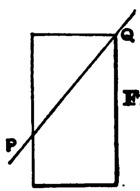



Fig. 172.

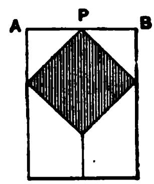



Fig. 173.

If a cube be cut by a plane containing the middle points of two top edges, two bottom edges, and two vertical edges, the section will be a *regular hexagon.*

If any square prism be cut by a plane, PQ, Fig. 172, containing an edge of either base, and oblique to the other base, the section will be a rectangle.

If the plane be passed through a corner, P, and parallel to a diagonal of either base, but oblique to the other base, the section will be a rhombus (Fig. 173).

Hexagonal and octagonal prisms cut in the same ways will produce symmetrical (143) elongated hexagons or octagons of various proportions.

Every kind of parallelogram, or other irregular plane figure bounded by straight lines, can be cut from oblique or otherwise irregular prisms.

**312.** The convex surface of any cylinder will be cut by any plane, cutting in a direction parallel to the axis, in *two parallel straight lines.*

The convex surface of any cone will be cut by any plane through the vertex in *two intersecting straight lines.*

All sections of a cylinder perpendicular to its axis are *equal circles.*

All sections of a cone perpendicular to its axis are *unequal circles*, that at the vertex being a *point.*

Thus, as solids of various kinds can be built up from plane figures, as before noticed, so, in turn, having given the few and simple elementary solids, all kinds of plane figures can be cut from them.

**313.** But we have a few other truths of form and position of another kind to mention, with which we will end this chapter. The line AB being tangent to the circle C at T, if all the points of the figure move equally in lines perpendicular to the paper, the line AB will generate a *plane* which will be tangent to the *cylinder* generated by the circle C at all points of the line generated by the point of contact, T.

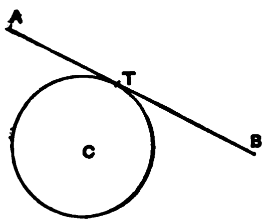


Fig. 174.

**314.** If any straight line drawn through the vertex, $v$, of a cone, Fig. 175, but so as not to intersect the cone, move parallel to itself, and touching the cone, it will generate a *plane* tangent to the cone at all points of the straight line, as $va$, described by the point $v$ of the moving line.

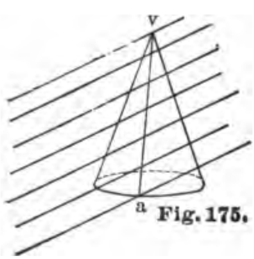


Fig. 175.

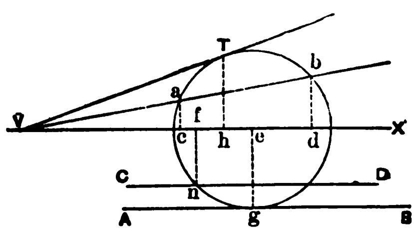


Fig. 176.

**315.** Again, let VT, Fig. 176, be a tangent to the circle *agb.* If both revolve together about an axis, VX, *oblique* to VT, the circle will generate a *sphere*, and VT a *cone* tangent to it at all points of a *circle* of radius T*h*, and perpendicular to the axis VX. If AB be *parallel* to VX, it will generate a *cylinder* tangent to the sphere on the *great circle* of radius, *ge*.

If V*ab*, oblique to VX, and CD, parallel to it, both revolve about VX with the circle, V*ab* will generate a *cone* which will intersect the sphere in circles of radii *ac* and *bd*, and CD will generate a cylinder which will intersect the sphere in two equal circles of radius equal to *nf*; and all these circles will be perpendicular to the common axis, VX, of all the bodies.

Likewise, if two intersecting circles, C and O, Fig. 177, or two intersecting lines, as VA and AB, Fig. 178, revolve together about a common axis, VX, the two circles will generate two spheres, which will intersect in a circle of diameter, *ab*, perpendicular to VX; and VA, Fig. 178, will generate a cone which will intersect the cylinder generated by AB in a circle of radius, AC, likewise perpendicular to VX.

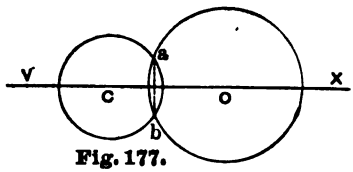


Fig. 177.

Finally, all surfaces, as that of a cylinder, cone, or sphere, formed by revolving any line about a fixed axis, are called surfaces of revolution; and from the foregoing illustrations we gather the general principle, that *all tangent or intersecting surfaces of revolution having a common axis, touch, or intersect, each other in circles perpendicular to that axis, and having their centres in it.*

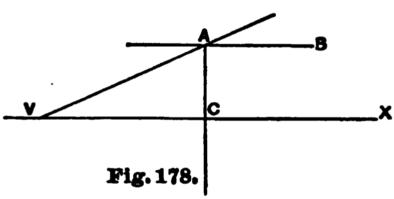


Fig. 178.

**Practice.**—1. Cut all the figures mentioned in Arts. 311, 312, from the solids there mentioned.

2. Lay a card against the surface of a sphere, and note the amount of its contact with the sphere.

3. Do the same for a cylinder and a cone.

# CHAPTER IX.

## PROJECTIONS OF ELEMENTARY SOLIDS.

**316.** We have seen, in all of the previous chapters about plane figures, that the figure of a triangle *is* of the same size and form as that triangle, the figure of a square *is* an equal square, and so on. This is because the figure is a flat figure having only two dimensions, and the paper on which it is drawn is a flat surface also having two dimensions. Hence the flat figure can be placed on the flat paper, and show itself there just as it is, as we have seen.

**317.** But the like is not true of the drawings of solid bodies; that is, no drawing of a cube is a cube, no figure of a cone is a cone: and the like is true of all drawings of solids. This is because solids have three dimensions, which cannot therefore all be shown in a single flat figure.

**318.** But we have shown in Chap. II. (Fig. 21), *first*, that two or more different figures of a solid can be made, each one of which will show two of its dimensions; and, *second*, that these figures can be arranged so that it can be understood which dimensions of the solid each figure shows.

**319.** The figures which thus represent solids so that we can understand their real forms, so as to be able to make the solids if necessary, are called *projections*. They are also called working-drawings, because they guide workmen in knowing what is meant to be made.

**320.** Working-drawings, or projections, thus form a kind of language. When we learn, by looking at them, what they represent, without seeing the object itself which they represent, we are said to read them.

The making and the reading of working-drawings is very important in connection with a great many useful arts and industries. We shall therefore, in this chapter, give some simple examples of such drawings.

**321.** Let us next see how to choose what examples we will take; for the number and the variety of objects of exact form furnished by all the arts and trades are so great, that we might hardly know what to choose, or where to begin.

Now, by looking about, or thinking awhile, we shall soon find that most common objects are prisms, pyramids, cylinders, cones, or spheres, or parts of them, or with slight additions to them, or are formed by combining two or more of them.

Hence, if we can make and read the projections, or working-drawings, of the elementary solids just mentioned, in various positions of them, we can make and read drawings of a multitude of objects which are composed of them. This we will now proceed to do.

### Prisms.

Knowing (284) that a cube has 6 equal square faces, A, Fig. 179, is the "plan" (83), or top view, of a cube, and B is its "elevation," or front view: that is, in looking down on the cube from above it, we see its upper face, A; and in standing before it, we see its front face, B. The sides of the squares A and B are each ¾ inch long. If we suppose Fig. 179 to be drawn on a scale of ¼ inch to 1 foot, it represents a cubical block, as of marble, 3 feet each way.

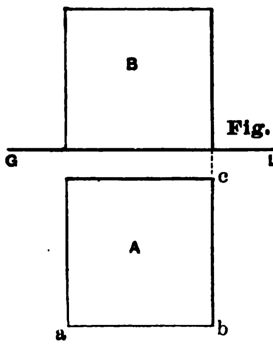


Fig. 179.

The line GL in this and all the following figures indicates the level on or above which the body is placed, and is called the *ground-line.*

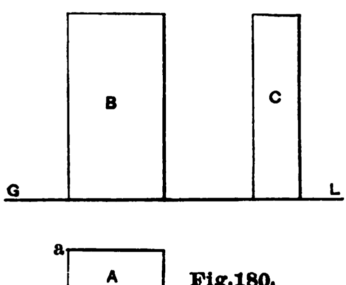


Fig. 180.

Fig. 180 represents the plan, A, the front view or elevation, B, and the side view or elevation, C, of an upright rectangular prism whose thickness, *ab*, is 1 foot (on a scale of ¼ inch to 1 foot), whose width, *bc*, is 2 feet, and whose height is 4 feet. The front face B stands upright on *bc*, and the face C stands at *ab*.

The six following are practical examples of the application of four-sided prisms: —

Fig. 181 is the plan, or top view, of a stone slab, ABCD, to cover the top of a chimney. At *ab* and *cd* are two square holes to fit the flues of the chimney. The figure being drawn on a scale of *one-half* an inch to a foot, the slab is 32 inches long and 20 inches wide; the holes are 8 inches square, 6 inches from the edge of the slab, and 4 inches apart. In the same Fig. 181, A′B′ is the elevation showing the thickness of the slab, which is 3 inches. The positions of the flue-holes, *a′b′* and *c′d′*, are shown by dotted lines. It is customary thus to show the position of hidden parts.

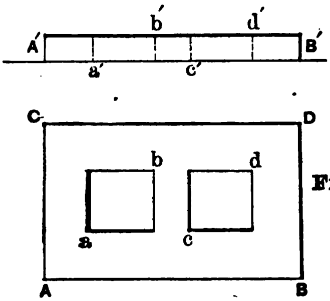



Fig. 181.

**322.** All of the examples here given should be drawn by the pupil much larger than can conveniently be here shown. Thus,

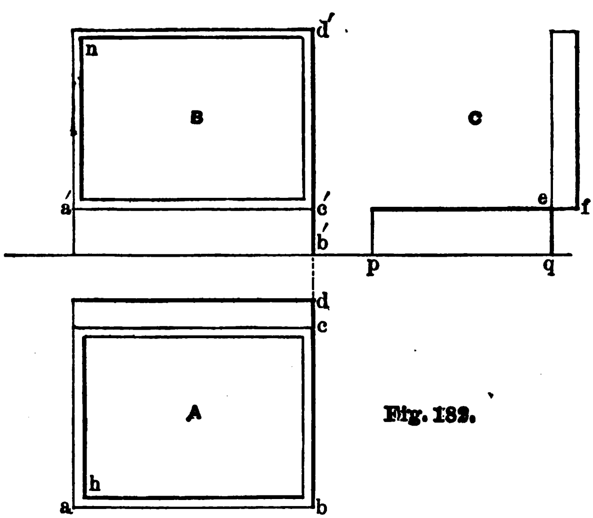



Fig. 182.

Fig. 181 may well be drawn on a scale of 1 or 2 or 3 inches to a foot; Fig. 182 may be drawn on a scale of ¼ or ½ or a whole inch to 1 inch; and likewise for other examples.

Fig. 182, drawn on a scale of *one-eighth* of an inch to one inch, represents a top view, A, a front view, B, and a side view, C, of a wooden work-box with the cover just half open. The length, *ab*, is 10 inches; the width, $bc = pq$, is $7\frac{1}{2}$ inches; the depth, $c'b'$, of the box is 2 inches, and that of the cover, $cd = ef$, is 1 inch. The thickness of the wood is $\frac{1}{4}$ inch.

323. The four figures now given illustrate *three rules*, which we will here state distinctly, so that they may be remembered in drawing all future figures.

*First.* — That elevation is placed directly over the plan which agrees with the plan with respect to the right and left of the observer. Thus, in Fig. 182 we see the length of the box from right to left, both in the plan, A, and the elevation, B, which is directly over it. Other elevations are placed on the same level with the first, and to the right or left of it as may be most convenient. Thus, C is the elevation of the box as it appears when one is facing the right-hand end, *bcd*, $b'c'd'$, of the box and cover.

*Second.* — Another rule relates to the position of the heavy lines, called *shade-lines*, which will be seen on the figures. The use of these lines is to assist in understanding, from such kind of drawings as projections are, what the real shape of the object represented is. These lines are supposed to divide those faces of the object which are in the light from those which are in the dark; and on all square-cornered bodies, placed in such positions as are shown in the last four figures, they are the *back and right-hand edges of the solid parts of the plan, and the lower and right-hand edges on the elevation.*

Thus, in Fig. 181 the lines which meet at *a* and at *c* in the plan are the back and right-hand edges of the solid stone around the flue-holes *ab* and *cd*. Likewise, CD and BD are the back and right-hand outer edges of the slab.

Again, in Fig. 182 the outer edges of the plan which meet at *d*, and the inner ones which meet at *h*, are back and right-hand edges of the wood of the box or of its cover. Likewise for the elevations: the edges of the cover which meet at *n*, and the lines $d'c'$ and $c'b'$, also the lines which meet at *f*, are right-hand and lower edges of the solid matter of the box and of its cover.

*Third.* — The third rule relates to the manner of lettering the projections so as to assist in understanding them. For this purpose, the same portions of the figure have the same letters in both projections, but with accents on the letters used on the elevation. Thus, in Fig. 182, *a* being the plan, and *a′* (*a* prime) the elevation of the front left-hand upper corner of the body of the box, we say the point *a*, *a′*, is that corner. Likewise *ab*, *a′b′*, is the front of the box; *bc*, *b′c′*, is its right-hand end, etc.

We will now proceed with some additional examples of bodies composed of prisms.

Fig. 183 represents a fragment of a brick arch or drain drawn on a scale of $\frac{1}{16}$ inch to 1 inch. The arch is not built with the bricks radiating from a centre, as in a true arch, but by the overlapping of the bricks, as shown in the elevation at *a′* and *b′*. This method of building a covered passage is interesting, because it was practised in very ancient times, before the real arch had been invented.

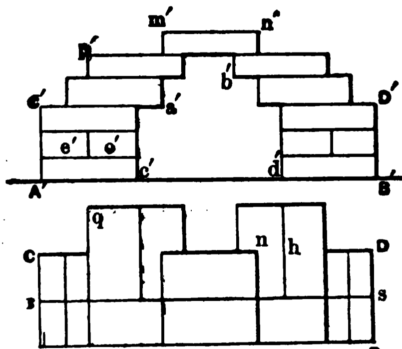


Fig. 183.

This figure is an example of those in which the elevation is more easily understood than the plan, and is therefore naturally made first. The width of the opening, *c′d′*, is 1 foot. All the bricks show their length, 8 inches, and thickness, 2 inches, except on both sides, as at *e′* and *o′*, where the ends, 4 inches by 2 inches, appear. Each overlapping brick, as *a′* and *b′*, overhangs the one below it 2 inches.

The top view, or plan, ABC*q*D, does not of itself show clearly what the structure is, as the elevation does, which shows the form of the end of the passage. But the plan is useful to show the arrangement of the bricks. Thus, the top brick, *m′n′*, is one of a row as long as the passage, and two of which are shown at *mn*. Again, to prevent a joint or seam, as *rs*, from extending entirely across the work at every four inches, some of the bricks in each layer, as those at *q* and *h*, lie across these joints, and "break" them, as it is said.

Fig. 184 is another simple example of brick-work. It shows the plan, ABCD, and elevation, A′B′, of one layer of bricks of a chimney 20 inches square, with one flue, *ab*, of 12 inches square. To bind the work well together, each corner brick lies alternately forwards and backwards, as the brick *cc'*, and from right to left, as the brick *dd'*, in the successive layers. The scale is here again $\frac{1}{16}$ inch to 1 inch.

The pupil may draw the figure on a scale of $\frac{1}{4}$ inch to 1 inch, and may show the layer beginning with the brick *d*, continued by corner B around to *b*.

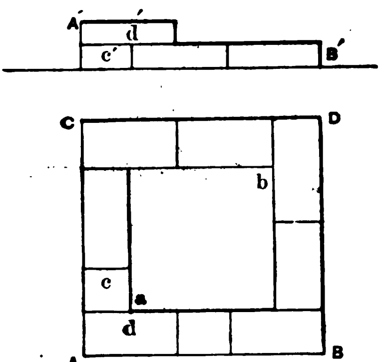



Fig.184.

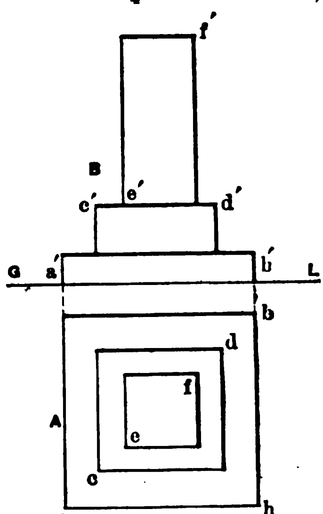



Fig.185.

Fig. 185 is the plan, A, and elevation, B, of a small stone monument, consisting of three parts, or members, each of which is a square prism, and is drawn on a scale of $\frac{1}{4}$ inch to 1 foot. The base, $ab - a'b'$, is 4 feet square and 8 inches thick; the plinth, $cd - c'd'$, is $2\frac{1}{2}$ feet square and 1 foot thick; and the shaft, or pillar, $ef - e'f'$, is $1\frac{1}{2}$ feet square and $3\frac{1}{2}$ feet high.

Note the location of the shade-lines according to the rule (2d) in Art. 323, and be careful to draw all the squares in the plan with the same centre.

**324.** By reviewing the figures thus far made in this chapter, it is easy to see that in all of them we could either have made the plan first, or the elevation first, as we pleased. The reason for this is, that the plan and the elevation each show two dimensions of the object represented. But sometimes the object to be drawn is so placed that one of its projections shows but one dimension in its real size. In such cases we always *must* make first that projection which shows two dimensions of the object in their real size. An example will make this clear.

Suppose a monument to consist of a block, *ab*, 2 feet square and 3 feet high, resting on a base, *de*, 4 feet square and 1½ feet thick, in the manner shown in Fig. 186. Unless we knew how to calculate the distance *ab*, that is, the diagonal, from the length of the side *ac*, we could not first draw the elevation, because there *a′b′* is equal to *ab*. To obtain the elevation purely by drawing operations, we take A, the centre of the base, as the centre of the block *ab*. From A we measure half the width of the block, or 1 foot, each way on each of the diagonals, as *de* of the base, and thus find four points, as *p* and *q*, in the sides of the plan of the block. Through these points draw the sides, as *ac* and *bc*, of the block parallel to the diagonals of the base. Having thus made the plan of the block, we find its elevation *a′b′c′* by drawing lines from *a*, *b*, and *c*, parallel to *df*; that is, perpendicular to the line GL, on which the elevation stands. This line is called the ground-line. The operation of finding one projection from the other is called projecting. We say that we project up the corners *a*, *b*, and *c* of the plan, to find the edges at *a′*, *b′*, and *c′* of the elevation of the block.

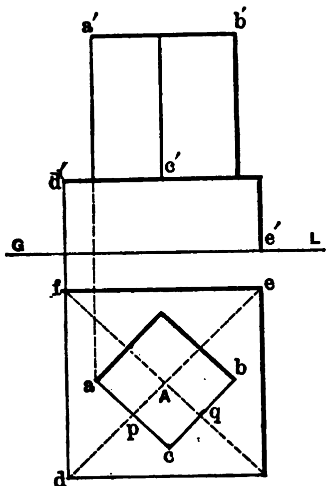


Fig. 186.

Fig. 187 shows the plan, A, and elevation, B, of a hexagonal prism. In this case a peculiar property of the regular hexagon (224) allows either projection of it to be made first when placed as in this figure: otherwise, the plan would have been made first; that is, *cd*, which is equal to *c′d′*, the width of the elevation of the right-hand face, *bc*, of the

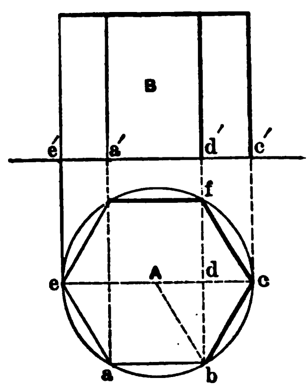


Fig. 187.

prism, is evidently equal to one-half of Ac. But Ac = *ab*; hence $cd = \frac{1}{2}ab$, or $c'd' = \frac{1}{2}a'd'$. The like is true of *ae*. Hence, if we know $a'd'$ = half an inch, we know that $c'd'$ and $a'e'$ each equal half an inch; and so we can make the elevation, B, first if we please.

The pupil can make a second elevation of this prism showing only two faces, as *bc* and *cf*.

Fig. 188 represents an octagonal prism each of whose sides is 3 feet high, and whose opposite sides are 4 feet apart, drawn on a scale of ¼ inch to a foot. Unless we know how to find, by calculation, how long $b'c'$ is as compared with *bc*, we cannot draw the elevation first, but must draw the plan first. A convenient way of drawing the plan, knowing that *ec* is 4 feet, is to make a square each side of which, as *mn*, is 4 feet, and then to proceed as in Art. 235. Having thus made the plan, project up the corners *e*, *a*, *b*, and *c* to find the vertical lines at $e'$, $a'$, $b'$, and $c'$ of the elevation, where only the middle face, A, which stands on *ab*, shows its real width. The side faces, B and C, standing on *ea* and *bc*, are, on the prism itself, of the same width as A, but, being seen obliquely, they do not appear as wide.

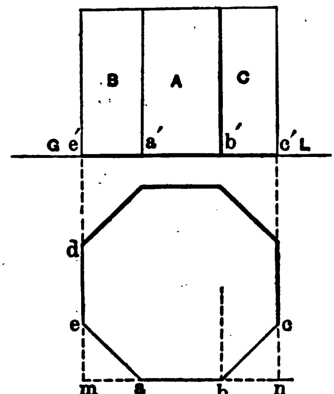


Fig. 188.

**Practice.**—1. In Fig. 179, draw the cube so that the side *ab* of the plan shall make an angle of 30° to the *right* with the ground-line GL. Which projection must then be made first, and what angle will *bc* make with the ground-line?

2. In Fig. 179, draw the cube so that *ab* shall make an angle of 30° to the *left* with the line GL.

3. In Fig. 180, draw A with *ab* parallel to GL, and then make the proper elevation over it.

4. In Fig. 180, draw A with its sides inclined to GL, and then make the corresponding elevation (see Fig. 186).

5. In Fig. 181, make the flues 8 inches by 12 inches, and make CD then 42 inches.

6. In Fig. 182, draw the plan A so that elevation C shall be the one properly over it, and corresponding with it.

7. In Fig. 183, put three lapping bricks, instead of two, on each side, and let each overlap the one below it an inch and a half.

8. In Fig. 184, draw the plan and elevation of a layer and a half of bricks of a chimney 16 inches by 36 inches, with two flues each 8 inches by 12 inches.

9. Draw Fig. 185 with *ah*, etc., making an angle of 45° with the ground-line GL.

10. In Fig. 185, replace the pillar, *ef* — *e'f'*, by a stone cross having its base at *ef* 1 foot square (on the same scale as Fig. 185), and let the cross be 4 feet high, with the horizontal arm 1 foot below the top, and 1 foot square on each side of the upright arm.

11. In Fig. 186, add an octagonal block, resting on the base *dfe* — *d'e'*, 1 foot thick, with its centre at A, and with its opposite sides 3½ feet apart, and two of them parallel to *df*.

12. Draw Fig. 187 as a monument by making the block A, B, stand on a base which shall be a triangular prism 1 foot thick, and with its plan an equilateral triangle one of whose sides shall be parallel to *bc* and 6 inches from it.

13. Draw Fig. 188 as a monument by making the prism stand on a base 1½ feet thick, and inscribed in a circle of 2½ feet radius, so that two corners shall be at the ends of the diameter parallel to GL.

14. On a square base 4 feet square and 1 foot thick, draw a block whose plan shall be a cross of four equal arms each 1 foot square, and let the height of the block be 6 feet, with a cube 1 foot every way on the centre of the top.

## Pyramids.

**325.** After what we have learned from Arts. 51 and 54, and others preceding them, about prisms and pyramids, and their relation to each other and to cylinders and cones, the following figures will be very easily understood and drawn. The edges which stand upright, at the corners of the base of a *prism* which stands on a level surface, all slant equally in a regular pyramid, and meet at one point over the centre of the base. Hence these edges can be seen both on the plan or top view, and on the elevation or front view, as we shall now see.

Fig. 189 is the top view, V — ABCD, and elevation, V'A'B', of a square pyramid drawn on a scale of 4 feet to 1 inch. It stands upright, and with a base, ABCD, 4 feet square, and is 6 feet high. VAB and V'A'B' are the top view and the front view of the front face of the pyramid. The other three faces are seen in the plan, equal to VAB, but not at all in the elevation. No face is seen in its real size. When we look directly at the end of a house so as to see the triangular end of the roof, we see by that how steep the roof is, or the distance from the top to the bottom of it. So, in Fig. 189, the line V'A' shows how steep the faces of the pyramid are, and also the true distance from the vertex, V, V', to the base on a line perpendicular to the side AC of the base. Such a line is shown at V*n* in the plan.

If, now, we were to cut out an isosceles triangle (142) of paper whose base should be equal to AC, and whose altitude should be equal to A'V', we should have a pattern showing the true size of each of the four equal faces of the pyramid.

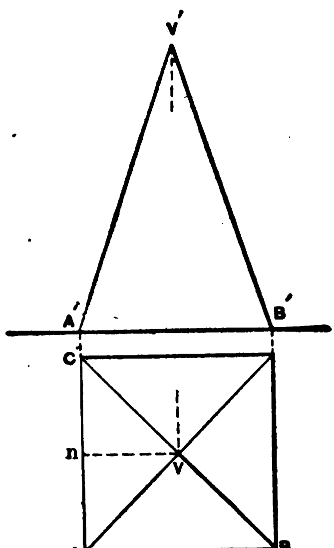



Fig. 189.

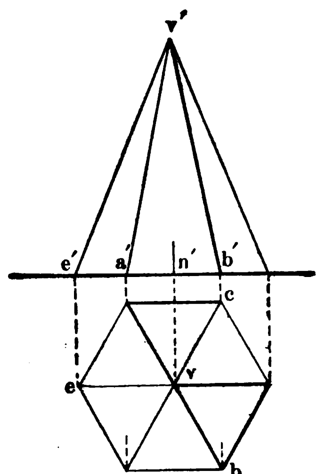



Fig. 190.

Fig. 190 represents a regular hexagonal pyramid drawn on a scale of 8 feet to the inch, and standing upright. Each side, as *ab*, of its base is 2 feet; and its vertical height, *v'n'*, is 10 feet. The plan shows the edges of the base and all the lateral faces, as *vab*, equally. In this and other like projections of pyramids, the pupil must not think of the plan as a flat figure, but must think of the point *v* as representing the vertex of the pyramid, and therefore as being at the height *v'n'* above the base *abc*.

As in the case of the square pyramid, no face shows in its real form or size, either in the plan or the elevation, though, as we shall next see, we can place the pyramid so that we can easily find this true size.

Fig. 191 shows the plan and the elevation of a regular hexagonal pyramid so placed that only two of its six faces, *vab* and *vbc*, can be seen in elevation, as shown at *v'a'b'* and *v'b'c'*. The face *vad* being seen exactly edgewise in the elevation on the line *a'v'*, we there see, as we did in Fig. 189 for the square pyramid, just how steep it is: hence, if we cut out an isosceles triangle whose base is equal to *ad*, and whose altitude is equal to *a'v'*, it will be the pattern of the true form and size of each one of the six equal faces of the hexagonal pyramid shown in Fig. 191.

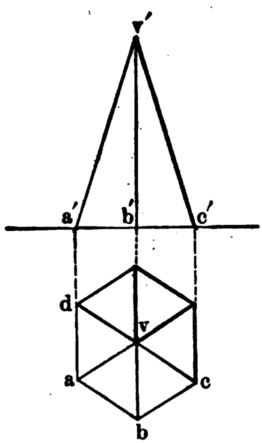


Fig. 191.

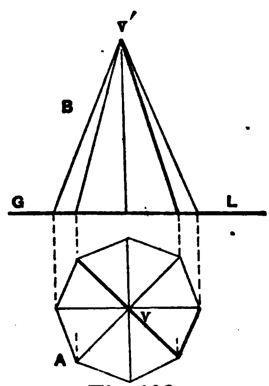


Fig. 192.

Fig. 192 is the plan, A, and elevation, B, of a regular octagonal pyramid, so placed that four of its sides are seen in the elevation. Its base is inscribed in a circle of 6 feet in diameter, drawn on a scale of 8 feet to 1 inch, and its height is 7 feet. No face is seen in its real size, either in plan or elevation. It is evident that no more than four faces of the pyramid can ever be seen in the elevation. No face is seen exactly edgewise in the elevation, and hence we cannot so well find the real size and form of any face as we can from the next figure.

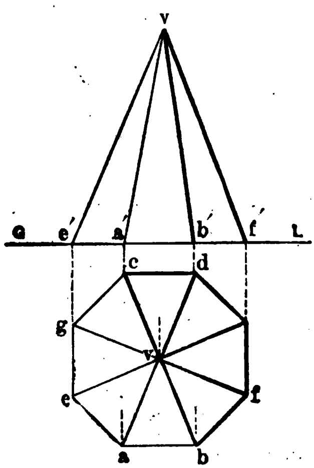


Fig. 193.

Fig. 193 is the plan and elevation of an upright regular octagonal pyramid drawn on a scale of $\frac{1}{8}$ inch to a foot, and so placed that two sides, *ab* and *cd*, of its base are parallel to the ground-line, GL. Each side of the base is 3 feet in length, and the height of the pyramid is 9 feet.

By comparing Figs. 190 and 193 we notice a general resemblance between the elevations; but, on examining them more carefully, we find that in Fig. 190 *e′a′* is just half of *a′b′* (Fig. 187), but that in Fig. 193 *e′a′* is more than half of *a′b′*. See also Fig. 188.

The face *veg* being seen edgewise at the line *v′e′* in the elevation, we see there the true perpendicular distance from its summit at *v*, *v′*, to the middle point of its base, *ge*. Hence, if we cut out a paper isosceles triangle, having its base equal to *ge* and altitude equal to *v′e′*, it will show the true form and size of each of the eight faces of the pyramid.

## Cylinders.

**326.** Having learned (51) that a cylinder differs from a prism only in the number of its sides, which is infinite, we can easily make the projections of a cylinder after having seen and understood those of a prism.

A cylinder is very often seen in steam-engines and other objects, and in various positions. Sometimes it stands upright, as the boiler of a steam fire-engine; sometimes we face the end of it, as when looking at the front end of a locomotive boiler; and sometimes it lies from right to left before us, as when a spool of cotton lies with its rounded surface on a table. We shall therefore principally notice the projections of these common positions of a cylinder.

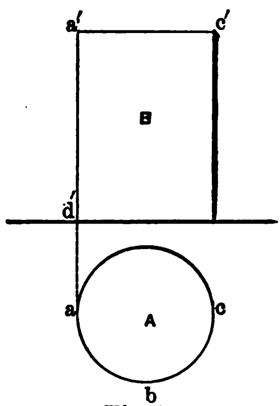


Fig. 194.

Fig. 194 is the plan, A, and elevation, B, of a cylinder $\frac{3}{4}$ inch in diameter and 1 inch high, and standing upright. The upper base, shown in the elevation at *a′c′*, is seen in the plan; and the front half of the convex surface, shown in the plan by the semicircle *abc*, is seen in the elevation. The elevation shows the length and diameter of the cylinder.

Every straight line, as $a'd'$, from the upper base to the lower base, lies in the convex surface of the cylinder, and is called an *element* of the surface.

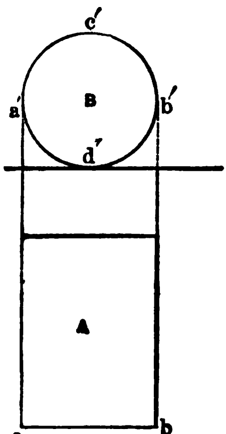


Fig. 195.

Fig. 195 is the plan, A, and elevation, B, of a cylinder placed so that the observer faces one end of it, $ab - a'b'c'd'$, as seen in the elevation. The upper half, shown at $a'c'b'$, is seen in the plan.

In Fig. 196, *abcd* is the plan of a cylinder lying horizontally from right to left. $m'n'o'p'$ is the elevation of the same cylinder. The circular plan in Fig. 194, and circular elevation in Fig. 195, enable us to know that the whole figure in each case represents a cylinder, and not a square prism. In Fig. 196, the plan and front elevation, both being rectangular, are the same as those of a square prism placed in the same manner. Hence an end view, $a''m''c''o''$, is added (see also Fig. 197), to make it certain that the figure represents a cylinder.

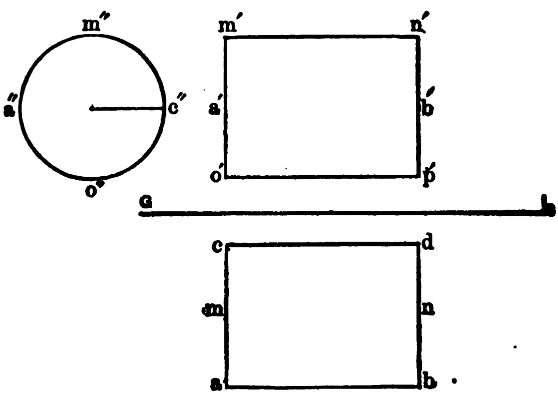


Fig. 196.

Fig. 197 shows the plan, A, the elevation, B, an a second elevation, C, of a cylinder, which is cut off obliquely, as shown by the slanting top $a'e'$, which is drawn at any angle we please.

The elevation B shows the cylinder as seen when looking in the direction of the arrow *c*. The elevation C shows the cylinder as seen when looking in the direction of the arrow *a*. The elevation

C is found from A and B by operations which we will next explain. As B and C represent the same cylinder standing on the same level, the positions of the same point, as seen on each, will be on the same horizontal line. Also, the distances of the points of C to the right or left of the central line $a''e''$ must be the same as the distances of the same points, as seen on A, to the right or left of the same line, *ae*, as shown in the top view, A. According to these two principles, we find $b''$ and $d''$, where horizontal lines from $b'$ and $d'$ meet the line $b''d''$, which is as far from $a''e''$ as *bd* is from *ae*. The construction being fully made, and the same point of the cylinder itself having the same letter on all three projections, as *c*, $c'$, $c''$, the figure shows that all the points of C are found as just described.

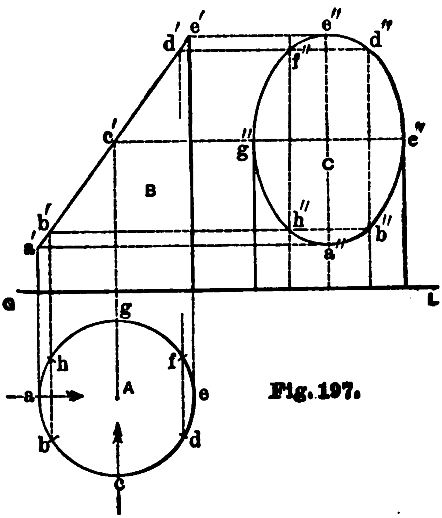



Fig. 197.

Fig. 198 shows the cylinder represented in Fig. 197, as if, it being made of paper, it were cut open at $aa'$, and unrolled so that its convex surface lies flat upon the paper. The length of GL is equal to the circumference of A (Fig. 197), and its divisions are taken from that circle. Then the heights of *b*, *c*, *d*, etc., above GL are the same as the heights of $b''$, $c''$, $d''$, etc., above GL in B, Fig. 197.

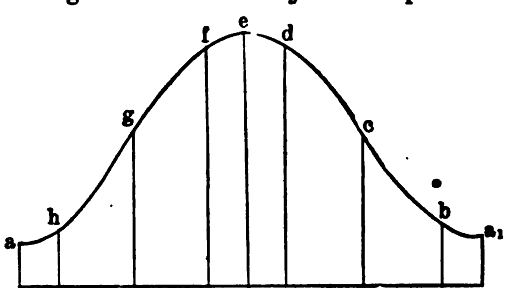



Fig. 198.

If, now, a paper pattern be cut out of the form of Fig. 198, and rolled up till $a_1$L joins *a*G, it will form the cylinder whose projections are shown in Fig. 197.

## Cones.

**327.** It would be so very easy to draw the projections of a cone in both of the positions shown in Figs. 195 and 196, that the work is left to the pupil, and we turn to some new variations of the exercise.

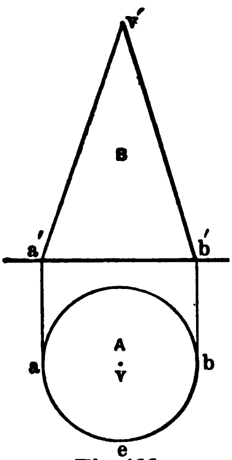



Fig. 199.

Fig. 199 represents a cone ¾ inch in diameter and 1¼ inches high, and standing upright. The plan, A, is a circle, and the elevation, B, is an isosceles triangle, showing the diameter *a′b′* of the base and the height of the cone.

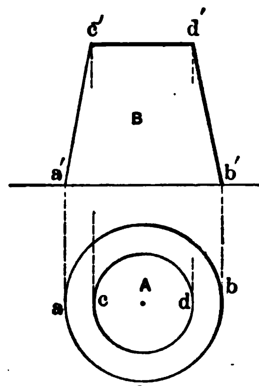



Fig. 200.

Fig. 200 represents the top view, A, and elevation, B, of the frustum of a cone; that is, the remainder of a cone after any portion containing the vertex has been cut off. This frustum is ¾ inch in diameter at the base *ab* – *a′b′*, and ½ inch in diameter at the upper base *cd* – *c′d′*, and is 3 feet high.

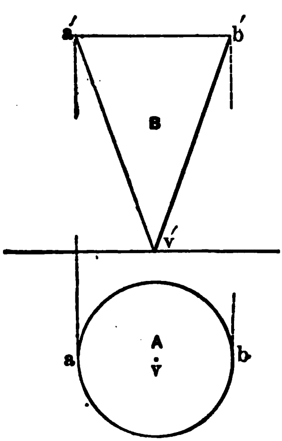



Fig. 201.

Fig. 201 shows the plan, A, and elevation, B, of an upside-down, or inverted, cone. The plan shows the surface of the base, and the elevation in this and the two preceding figures shows the front half of the convex surface.

In Fig. 199 the whole, and in Fig. 201 none, of the convex surface is seen in the plan.

**Practice.** — Draw Fig. 200 as if inverted, hollow, and of wood one-eighth of an inch thick.

### The Sphere.

**328.** We have naturally said, "prisms," "cylinders," etc., in the previous portions of this chapter, because these bodies may differ in form as well as size; but we naturally say "the sphere," for there is but one sphere as to form; different spheres are only different sizes of the same thing.

Since a sphere is circular in every direction, from every point upon it, it appears of a circular form in whatever way it is viewed. Hence, as shown in Fig. 202, the plan, C, and elevation, C′, of a sphere are two equal circles, with their centres, C and C′, on the same line, CC′, perpendicular to the ground-line.

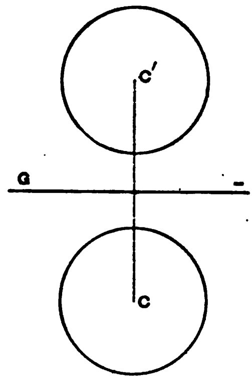


Fig. 202.

The principal difficulty with the projections of a sphere is in understanding the projections of portions of it in different positions. Remembering that different heights, or upper and lower, are seen in elevations, and the difference between front and back in plans, the next four figures will enable the pupil to draw others.

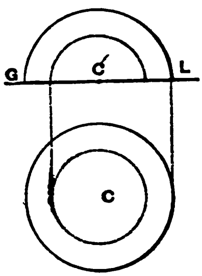


Fig. 203.

Fig. 203 shows the *upper* half of a hollow sphere $\frac{3}{4}$ inch in diameter and $\frac{1}{8}$ inch thick, resting on the level surface, as a table-top, indicated by the ground or base line GL.

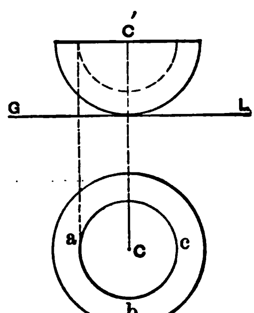


Fig. 204.

Fig. 204 shows the *lower* half of the same hollow sphere (Fig. 203). In Fig. 203, the inside edge is invisible in the top view, C, and front view, C′, and hence is dotted. In Fig. 204, the inner edge is visible in the plan, at *abc*. Fig. 203 is like a bowl upside down, and Fig. 204 like the same bowl right side up.

Fig. 205 represents the *back* half of the same hollow sphere again. The inner edge *aCb* is invisible in the plan, and visible in the front view, or elevation.

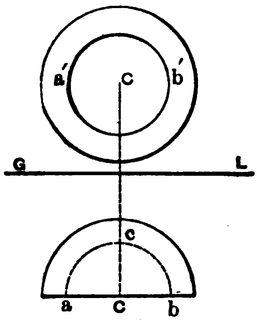



Fig. 205.

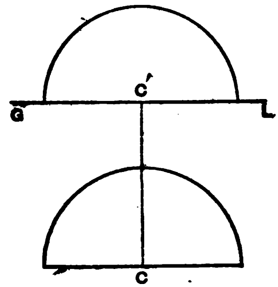



Fig. 206.

Finally, Fig. 206 shows the plan, C, and elevation, C′, of the upper back quarter of a sphere.

### Compound Forms.

**329.** After prisms, so many more bodies are in practice, composed of combinations of two or more of the preceding forms, than of bodies of the same kind, that we have placed the further practical examples here.

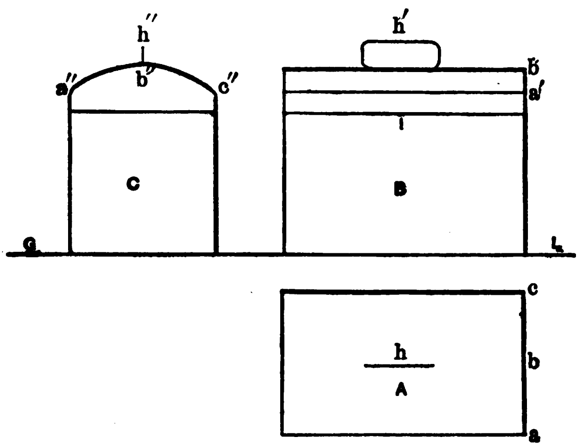



Fig. 207.

Fig. 207 shows the plan, A, the front elevation, B, and the end elevation, C, of a tin trunk 10 inches long, 6 inches wide, and 8 inches deep, drawn on a scale of $\frac{1}{8}$ inch to 1 inch. The body of the trunk is a rectangular, that is, a square-cornered, prism. The top, $abc - a'b' - a''b''c''$, of the cover is a cylindrical surface, on which is the wire handle $h$, $h'$, $h''$.

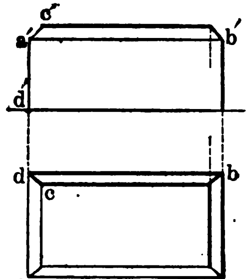


Fig. 208.

Figs. 208, 209, 210, are given to show that sometimes the plan, and sometimes the elevation, may be the same for different bodies, but that, generally, two projections together show fully the shape of the body represented by them.

Thus, Fig. 208 represents a stone block, consisting of a rectangular prism, with its top edge bevelled; that is, cut off obliquely. Speaking geometrically, the block is composed of a rectangular prism, $abd - a'b'd'$, and a thin frustum, $acb - a'b'c'$, of a rectangular pyramid.

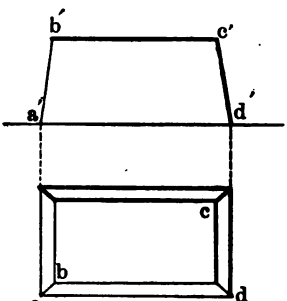


Fig. 209.

In Fig. 209, in which the plan is the same as in Fig. 208, the elevation shows that the entire block consists of the frustum, $abcd - a'b'c'd'$, of a rectangular pyramid.

In Fig. 210, the elevation is the same as if the block were only a rectangular prism, but the plan shows that it is a prism with rounded corners; that is, each of the four corners is a quarter-cylinder.

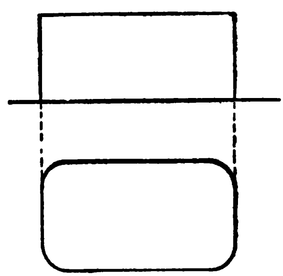


Fig. 210.

All the figures are drawn on a scale of 4 feet to 1 inch, and the block is 4 feet long at bottom, 3 feet 6 inches at top, 2 feet 3 inches wide, and 1 foot 9 inches high.

**Practice.** — 1. Draw Fig. 207 to represent the cover as just half open.

2. Draw Fig. 210 with a panel sunk 3 inches into the front. Draw it with a tablet projecting 3 inches from the front.

3. Make three projections, as in Fig. 207, of a flatiron lying on its face.

4. The same, but standing on its broad end.

Fig. 211 shows the plan and the elevation of a square iron nut, $abc$ — $a'b'c'$, and cylindrical bolt, $d$, $d'e'$, such as are used in fastening (bolting) together pieces of wood or iron.

Fig. 212 is the plan and elevation of a monument em-

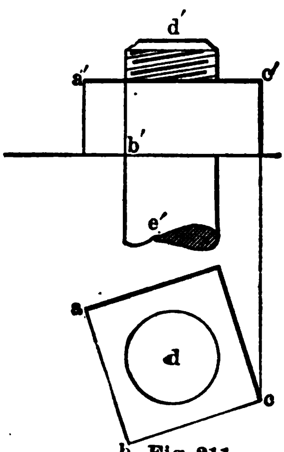


Fig. 211.

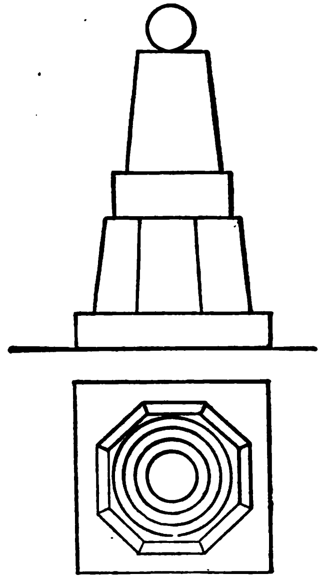


Fig. 212.

bracing an example of each of the five simple solids. The lowest portion is a square prism. Next comes a frustum of an octagonal pyramid, then a short cylinder, then the frustum of a cone, and, last, a sphere.

**330.** Other simple examples could be multiplied indefinitely, consisting of wood and stone posts, monuments, the principal lines, omitting details and ornaments, of buildings, bay-windows, doors, spires, dormer-windows, tables, stools, desks, fireplaces, and a great variety of carpentry joints, as seen in house-frames. The few following will serve to suggest others. As far as convenient to the pupil, let them, or others like them, be taken from objects within his reach, which he can measure, and draw to scale from his own measurement.

**Practice.**—1. Draw the projections of a fence-post of wood, stone, or iron.

2. Make the projections of a lamp-post.

3. Of a water hydrant.

4. Of a hand-sled (plan, side, and end elevations).
5. Of a water-faucet (three projections, as in 4).
6. Of a simple straight-armed gas bracket.
7. Of some plain book-shelves.
8. Of a barrel, firkin, or cask.
9. Of a bottle or a canister.
10. Of a tin tunnel or steamer.
11. Of any one or more floor, wall, or roof joints in the timber framing of any house that may be building near you.
12. Of a wooden table (three projections).
13. Of a simple desk (three projections).
14. Of a short flight of wood or iron steps, with or without side piers.
15. Of a door, gate, or blind.
16. Of a single door-hinge.
17. Of a plain marble mantel-piece.
18. Of a flower-pot and saucer.
19. Of a fragment of a railroad rail (three projections).
20. Of a plain chest of drawers.
21. Of a small hennery or chicken-coop.
22. Of a saw-horse or a wheelbarrow.
23. Of a mason's hammer (three projections).
24. Of a mucilage bottle or an inkstand.
25. Of a dictionary or other large book.
26. Of a harrow or a tip-cart.
27. Of a prismatic or cylindrical pump (three projections).
28. Of a stone or wood watering-trough.
29. Of a foot-stool or a bench.
30. Of a wooden chair, all in straight pieces (three projections).
31. In Fig. 189, cut off the pyramid, horizontally, 4 feet above its base, and place it upon a plinth 4½ feet square and 9 inches thick, resting on a base 5 feet square and 1 foot thick.
32. Convert Fig. 190 into the drawing of the upper part of a small spire, by making the pyramid 25 feet high, and standing on a hexagonal prism 3 feet on each side and 12 feet high, which in turn rests on a cube 9 feet square.
33. Likewise convert Fig. 193.
34. Convert Fig. 194 into a monument, by adding below the cylinder an octagonal base of suitable size, and above it the frustum of a cone, and on that a hexagonal prism; and make the height of the cylinder equal to half its diameter.
35. Endeavor, by the help of Fig. 196 (see also the explanation of Fig. 197), to make the elevation of a cylinder placed in a horizontal position, as in Fig. 196, only that its plan shall be inclined at an angle of 30° to GL (see also Figs. 186 and 211).
36. Make the development (Fig. 159) of the cone, and of the frustum shown in Figs. 199 and 200.
37. Make the plan and elevation of the *right* or the *left hand* half of the hollow sphere shown in Fig. 203.

38. Make the plan and elevation of each of the other three quarters of the sphere, one of which is shown in Fig. 206.

39. In Fig. 207, replace the cylindrical top of the trunk by a thin frustum of a four-sided pyramid (see Fig. 208).

40. In place of Fig. 208, take a rectangular prism, and then cut off each corner so as to leave it a triangle whose vertex shall be a corner, as *dd'*, of the base of the block, and whose base shall be a line through *c*, making an angle of 45° with *ad* and *bd*.

41. In like manner make the corners of Fig. 210 quarter-cones instead of quarter-cylinders.

42. In Fig. 211, change the position of the nut, or make it hexagonal.

**331.** Any one who has access to furniture in the Eastlake style, that is, that which is mostly made with straight-line work, can find a large range of additional examples, consisting of bedsteads, chairs, tables, bookcases, bureaus, etc.; while others who have access to carpenters' or machinists' shops can find a great variety of other examples.